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Anatomy of bubbling solutions

Kostas Skenderis and Marika Taylor

Institute for Theoretical Physics, University of Amsterdam,
Valckenierstraat 65, 1018XE Amsterdam, The Netherlands
E-mail: skenderi@science.uva.nl, taylor@science.uva.nl

Abstract: We present a comprehensive analysis of holography for the bubbling solutions of Lin-Lunin-Maldacena. These solutions are uniquely determined by a coloring of a 2-plane, which was argued to correspond to the phase space of free fermions. We show that in general this phase space distribution does not determine fully the 1/2 BPS state of $N = 4$ SYM that the gravitational solution is dual to, but it does determine it enough so that vevs of all single trace 1/2 BPS operators in that state are uniquely determined to leading order in the large N limit. These are precisely the vevs encoded in the asymptotics of the LLM solutions. We extract these vevs for operators up to dimension 4 using holographic renormalization and KK holography and show exact agreement with the field theory expressions.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence.

Contents

1. Introduction

Supersymmetric supergravity solutions play an important role in developing our understanding of holographic dualities. Such solutions provide valuable examples where one can carry out detailed computations and, using non-renormalization properties, make quantitative tests of gravity/gauge dualities at and away from conformal fixed points. Given the precise holographic dictionary available in such cases, one may also understand in detail how the spacetime is reconstructed from gauge theory data. One would hope to take from these examples generally applicable methods and principles, along with insight into the inner workings of holography.

Supergravity solutions that asymptote to $AdS_5 \times S^5$ describe either a deformation of $N = 4$ SYM or the theory in a non-trivial state. The most supersymmetric non-trivial vacua of $N = 4$ SYM theory preserve 16 supersymmetries. In this context it is interesting to consider the SYM theory on both Minkowski spacetime $R^{(1,3)}$, in which case we have $N = 4$ on the Coulomb branch, and the theory on $R \times S^3$. These two cases are equivalent in the conformal vacuum since the two backgrounds are mapped to each other by a Weyl transformation but differ on a generic half supersymmetric state (which spontaneously breaks the conformal invariance of the $N = 4$ SYM theory). Of course, the two theories are still related in the decompactification limit of S^3.

Using standard D-brane physics, one expects that the holographic dual of $N = 4$ SYM on the Coulomb branch is the near-horizon limit of multi-center D3 brane solutions [1]. The Coulomb branch may be parametrized by the vevs of chiral primary operators and the gravity/gauge theory duality together with non-renormalization theorems imply that the vevs computed at weak coupling are non-renormalized and must therefore also be reproduced by the holographic computation. In [2], building on [3, 4], we indeed succeeded in extracting these vevs from a generic multi-center solution, showing exact agreement with field theory. This provides a highly non-trivial test of the correspondence away from the conformal point – an infinite number of vevs was quantitatively matched – and also illustrates the maturity of holographic methods as it shows that one can go beyond qualitative matching, performing precise quantitative computations.

Supergravity solutions corresponding to $N = 4$ SYM on $R \times S^3$ were recently constructed in [5]. The solutions of [5] preserve an $R \times \mathrm{SO}(4) \times \mathrm{SO}(4)$ bosonic symmetry and 16 supersymmetries. These "bubbling solutions" are uniquely determined by a coloring of the 2-plane into black and white regions. Based on earlier work relating 1/2 BPS states to free fermions [6, 7], this distribution was argued to map to the phase space distribution of free fermions and supporting evidence was provided in [8 – 11]; see also [12 – 17].

It is often stated in the literature that the gauge theory dual of the bubbling solution is the matrix model associated with free fermions. One of the aims of this work is to understand to which extent this assertion is valid by applying standard AdS/CFT methods. That is, we will address the question of whether the matrix model captures the entire vacuum structure. In the AdS/CFT correspondence the asymptotics of the supergravity solution encode QFT data. In particular the vacuum structure of the dual QFT can be extracted from the near boundary asymptotics of the solution. In the first part of this paper we will use holographic renormalization [18] and KK holography [4] in order to extract the vevs from the solutions of [5]. Any proposal for the field theory dual must reproduce these results.[1]

Let us now discuss the QFT side. By the operator-state correspondence, all 1/2 BPS states of $N = 4$ SYM can be obtained by acting with 1/2 BPS operators on the conformal vacuum. The operators constructed from the 6 scalars X^m of $N = 4$ SYM lie in the $(0, l, 0)$ representation of the SU(4) R-symmetry (see, for example, the review [19]). Up to a $\mathrm{U}(3) \subset \mathrm{SU}(4)$ rotation, every such operator can be represented holomorphically using a single complex combination of the scalars $Z = X^1 + iX^2$. Thus gauge invariant operators built from these scalars preserve an SO(4) part of the SU(4) R-symmetry and have a definite SO(2) charge j under rotations in the X^1–X^2 plane. A convenient basis for these operators is the Schur polynomial basis [6] and an arbitrary half BPS state $|\Phi\rangle$ preserving SO(4) R symmetry can be written as a superposition of states

$$|\Phi\rangle = \sum_R a_R \chi_R(Z)|\Omega\rangle \equiv \mathcal{O}_\Phi|\Omega\rangle \tag{1.1}$$

for suitable complex coefficients a_R, where $\chi_R(Z)$ is the Schur polynomial associated with the[2] $\mathrm{U}(N)$ representation R and $|\Omega\rangle$ is the conformal vacuum. The representation R may be labeled by a Young tableau and the associated Schur polynomial $\chi_R(Z)$ has degree equal to the number of boxes n and in general involves both single and multi-trace contributions. Thus the operator $\mathcal{O}_\Phi$ is equal to a sum of terms each of which has dimension equal to charge, $\Delta = j = n$, for any $n > 1$. It follows that in order to specify the theory we need to supply the coefficients a_R and any gravitational dual should encode these coefficients.

When the field theory is formulated on $R \times S^3$ one may reduce over the S^3 to obtain a one-dimensional model involving an infinite number of fields (KK modes). Given the large amount of supersymmetry, however, one might anticipate that the vacuum structure, i.e. the coefficients a_R in (1.1), may be encoded, at least to leading order in the large N limit, in the truncation of the S^3 reduction to only the s-mode of the complex scalar Z. We will take this as a working assumption in this paper. It would be interesting to investigate the validity of this assertion in general. We should emphasize that one should be very cautious about using properties of this matrix model which are subleading in N (to infer properties of the dual spacetimes etc.) since these are likely to be different from the true $1/N$ corrections of $N = 4$ SYM on $R \times S^3$.

[1]Here we assume that the 16 supercharges protect the vevs from acquiring quantum corrections, as in the case of $N = 4$ SYM on $R^{(1,3)}$.

[2]We are really interested in $\mathrm{SU}(N)$ gauge theory but the difference between $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ is subleading in the large N limit.

Standard arguments map the matrix model to free fermions, whose phase space distribution is meant to correspond to the coloring of the 2-plane that determines the LLM solution. In the fermion picture, we have N free fermions with the ground state being the completely filled Dirac sea. This state corresponds to $AdS_5 \times S^5$. Excited states are in one to one correspondence with the Schur polynomials, the excitation numbers being directly determined from the the length of the rows of the associated Young tableaux. A generic excited state in the free fermion picture is then in direct correspondence with the 1/2 BPS state (1.1) of $N = 4$ SYM. The crucial question is then: does a phase space distribution for the fermions uniquely determine the state $|\Phi\rangle$? We show that this is *not* the case, i.e. the phase space distribution does not determine all coefficients a_R, but it does determine the state enough so that the vevs of *all* (single trace) *chiral primaries* in this state are uniquely determined! These are precisely the vevs encoded in the asymptotics of the supergravity solution.

The results of the holographic computation show that the LLM solutions (generically) encode vevs of *all* SO(4) singlet operators; such operators can be labeled by their SO(2) charge j. So to check the correspondence one should compute these vevs in the field theory. Here we face the first obstacle. While maximally charged operators (those whose SO(2) charge is equal in magnitude to the dimension) involve only the Z field and so can be implemented in the matrix model [10], all other operators involve all six scalars and thus appear to involve fields not included in the matrix model.

To see how one can deal with this issue, recall that the 1-point function of an operator $\mathcal{O}$ in the state $|\Phi\rangle$ is equivalent to the 3-point function in the conformal vacuum between $\mathcal{O}$, the operator that creates $|\Phi\rangle$ from $|\Omega\rangle$ and its conjugate,

$$\langle \mathcal{O} \rangle_\Phi = \langle \Omega | \mathcal{O}_\Phi^\dagger \mathcal{O} \mathcal{O}_\Phi | \Omega \rangle. \tag{1.2}$$

Suppose this correlator is computed in free field theory. Since $\mathcal{O}_\Phi$ is constructed only from Z the 3-point function receives contributions only from part of $\mathcal{O}$ that contains Z. Thus for the free field computation of the 1-point functions in (1.2) one may set to zero all fields but Z (and $\bar{Z}$) in the chiral primary operators. We emphasize, however, that this truncation would in general give incorrect answers if used for different computations, e.g. two point functions. If the free field computation were to be renormalized, then fields apart from Z would of course contribute in loops. However, three-point functions of single trace chiral primary operators of $N = 4$ SYM are known not to renormalize [20] and it is believed that three point functions of protected multi-trace operators are similarly non-renormalized [21]. We indeed find that the vevs computed using free field results for multi trace operators do agree with those extracted holographically, thus confirming the expectation of non-renormalization.

The truncated operators can therefore be implemented in the matrix model. We do this explicitly for all operators up to dimension four (whose vevs we also extract from the gravity solutions). In particular, we show that each of these operators can be expressed as linear combinations of bilinears of fermion creation and annihilation operators. The coefficients in the linear combinations are fixed by demanding that the operators have zero expectation values at the conformal vacuum, 3-point functions with single trace operators are correctly

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reproduced, and the vev of the operators have the correct limit in the decompactification limit of the S^3. In this limit the phase space distribution maps to the distribution of eigenvalues of the scalars in the Coulomb branch of $N = 4$ SYM. Having implemented the operators in the matrix model it is then straightforward to compute their vevs in a general state $|\Phi\rangle$ and we find exact agreement with the holographic computations!

This paper is organized as follows. In the next section we summarize how to extract holographic data from asymptotically $AdS_5 \times S^5$ solutions. The resulting expressions for the holographic vevs in terms of supergravity field asymptotics are applicable not just to the bubbling solutions of interest in this paper, but to more general 1/4 and 1/8 BPS bubbling solutions. In section 3 we review the LLM solutions and extract the vevs of all maximally charged operators and of all operators with any charge up to dimension four. In section 4 we discuss the dual description of the bubbling solutions. We show what information about the state is captured by the distribution and hence the gravity solution; we reproduce the holographic vevs in section 5 and we explicitly match certain specific symmetric distributions with 1/2 BPS states in section 6. In section 7 we discuss our results.

Appendices A and B review relevant properties of spherical harmonics and of scalar chiral primary operators in $\mathcal{N} = 4$ SYM whilst appendix C discusses the large N scaling of three point functions. Appendix D is rather tangential to the focus of the paper: we discuss the Killing spinors for the LLM supergravity solutions. These were discussed in [5] but only half of them were correctly identified and they are missing local phase factors which (drop out of the fermion bilinears used to construct the supergravity solution but which) are needed to solve the Killing spinor equations.

2. Extracting holographic data

In this section we will give a self-contained summary of the method of Kaluza-Klein holography, developed in [4], which allows the computation of all 1-point functions from any asymptotically $AdS_p \times X_q$ supergravity solution.

The basic steps in this method are the following. First one expresses the deviation of the supergravity solution from $AdS_p \times X_q$ in terms of the complete basis of harmonics of the compact manifold X_q; let the expansion coefficients be denoted collectively as $\psi^{\mathcal{I}}$. Now one forms gauge invariant combinations of these fluctuations, $\hat{\psi}^{\mathcal{I}}$, that satisfy field equations which can be expanded perturbatively in the number of fluctuation fields. Schematically these field equations may be written

$$\mathcal{L}_{\mathcal{I}}\hat{\psi}^{\mathcal{I}} = \mathcal{L}_{\mathcal{I}\mathcal{J}\mathcal{K}}\hat{\psi}^{\mathcal{J}}\hat{\psi}^{\mathcal{K}} + \mathcal{L}_{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{L}}\hat{\psi}^{\mathcal{J}}\hat{\psi}^{\mathcal{K}}\hat{\psi}^{\mathcal{L}} + \cdots, \tag{2.1}$$

where $\mathcal{L}_{\mathcal{I}_1\cdots\mathcal{I}_n}$ is an appropriate differential operator. Since $\mathcal{L}_{\mathcal{I}_1\cdots\mathcal{I}_n}$ involves derivatives, the set of field equations cannot generically be integrated into an action. However, one can always define p-dimensional fields $\Psi^{\mathcal{I}}$ by a non-linear Kaluza-Klein reduction map of the fields $\psi^{\mathcal{I}}$:

$$\Psi^{\mathcal{I}} = \psi^{\mathcal{I}} + \mathcal{K}^{I}_{\mathcal{J}\mathcal{K}}\psi^{\mathcal{J}}\psi^{\mathcal{K}} + \cdots, \tag{2.2}$$

where $\mathcal{K}^I_{\mathcal{J}\mathcal{K}}$ contains appropriate derivatives. The reduction map is such that the fields $\Psi^{\mathcal{I}}$ do satisfy field equations which can be integrated into an action. Given this p-dimensional action, it is then straightforward to obtain the one point functions of operators in terms of the asymptotics of the fields $\Psi^{\mathcal{I}}$, using the well-developed techniques of holographic renormalization [22 – 29]; for a review, see [18]. We will now give the details of each step in the case of interest.

Let us consider any asymptotically $AdS_5 \times S^5$ solution of type IIB, which involves only the metric and 5-form field strength. (It is straightforward to include all other fields of type IIB, but unnecessary for our application here to the LLM bubbling solutions.) The IIB SUGRA field equations[3] for the metric and 5-form field strength are given by:

$$R_{MN} = \frac{1}{6} F_{MPQRS} F_N{}^{PQRS}, \qquad F = *F. \tag{2.3}$$

These equations admit an $AdS_5 \times S^5$ solution

$$\begin{aligned} ds_o^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2} dx_{||}^2 + d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\phi^2 \\ F^o_{\mu\nu\rho\sigma\tau} &= \epsilon_{\mu\nu\rho\sigma\tau}, \qquad F^o_{abcde} = \epsilon_{abcde}. \end{aligned} \tag{2.4}$$

where (μ,ν) and (a,b) denote AdS_5 and S^5 indices respectively; $M, N, \ldots$ are $10d$ indices whilst x denotes AdS coordinates and y denotes S^5 coordinates. We will consider here solutions that are deformations of $AdS_5 \times S^5$ such that

$$\begin{aligned} g_{MN} &= g^o_{MN} + h_{MN}, \\ F_{MNPQR} &= F^o_{MNPQR} + f_{MNPQR}. \end{aligned} \tag{2.5}$$

These fluctuations can be expanded in spherical harmonics as:

$$\begin{aligned} h_{\mu\nu}(x,y) &= \sum h^{I_1}_{\mu\nu}(x) Y^{I_1}(y); \\ h_{\mu a}(x,y) &= \sum (B^{I_5}_{(v)\mu}(x) Y^{I_5}_a(y) + B^{I_1}_{(s)\mu}(x) D_a Y^{I_1}(y)); \\ h_{(ab)}(x,y) &= \sum (\hat{\phi}^{I_{14}}_{(t)}(x) Y^{I_{14}}_{(ab)}(y) + \phi^{I_5}_{(v)}(x) D_{(a} Y^{I_5}_{b)}(y) + \phi^{I_1}_{(s)}(x) D_{(a} D_{b)} Y^{I_1}(y)); \\ h^a_a(x,y) &= \sum \pi^{I_1}(x) Y^{I_1}(y), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} f_{\mu\nu\rho\sigma\tau}(x,y) &= \sum 5 D_{[\mu} b^{I_1}_{\nu\rho\sigma\tau]}(x) Y^{I_1}(y); \\ f_{a\mu\nu\rho\sigma}(x,y) &= \sum (b^{I_1}_{\mu\nu\rho\sigma}(x) D_a Y^{I_1}(y) + 4 D_{[\mu} b^{I_5}_{\nu\rho\sigma]}(x) Y^{I_5}_a(y)); \\ f_{ab\mu\nu\rho}(x,y) &= \sum (3 D_{[\mu} b^{I_{10}}_{\nu\rho]}(x) Y^{I_{10}}_{[ab]}(y) - 2 b^{I_5}_{\mu\nu\rho}(x) D_{[a} Y^{I_5}_{b]}(y)); \\ f_{abc\mu\nu}(x,y) &= \sum (2 D_{[\mu} b^{I_5}_{\nu]}(x) \epsilon_{abc}{}^{de} D_d Y^{I_5}_e(y) + 3 b^{I_{10}}_{\mu\nu}(x) D_{[a} Y^{I_{10}}_{bc]}(y)); \\ f_{abcd\mu}(x,y) &= \sum (D_\mu b^{I_1}_{(s)}(x) \epsilon_{abcd}{}^e D_e Y^{I_1}(y) + (\Lambda^{I_5} - 4) b^{I_5}_\mu(x) \epsilon_{abcd}{}^e Y^{I_5}_e(y)) \\ f_{abcde}(x,y) &= \sum b^{I_1}_{(s)}(x) \Lambda^{I_1} \epsilon_{abcde} Y^{I_1}(y); \end{aligned} \tag{2.7}$$

[3]The field strength differs by a factor of 4 from the conventions in [30].

Numerical constants in these expressions are inserted so as to match with the conventions of [31]. Parentheses denote a symmetric traceless combination (i.e. $A_{(ab)} = 1/2(A_{ab} + A_{ba}) - 1/5 g_{ab} A^a_a$). $Y^{I_1}, Y^{I_5}_a, Y^{I_{14}}_{(ab)}$ and $Y^{I_{10}}_{[ab]}$ denote scalar, vector and tensor harmonics whilst Λ^{I_1} and Λ^{I_5} are the eigenvalues of the scalar and vector harmonics under (minus) the d'Alembertian. The subscripts t, v and s denote whether the field is associated with tensor, vector or scalar harmonics respectively, whilst the superscript of the harmonic label I_n derives from the number of components n of the harmonic. Relevant properties of the spherical harmonics are summarized in appendix A.

In what follows it will be useful to label perturbations by both the degree k of the associated harmonic and by the degeneracy of such harmonics. For example, π^{kI} will denote the fluctuations associated with degree k scalar harmonics with I labeling the SO(6) quantum numbers.

2.1 Gauge invariant quantities

When computing the spectrum it is useful to impose the de Donder-Lorentz gauge choice, as in [31], which imposes the following conditions on the metric fluctuations

$$D^a h_{(ab)} = D^a h_{a\mu} = 0, \tag{2.8}$$

along with analogous conditions on the five-form fluctuations. These gauge conditions remove terms involving gradients of spherical harmonics.

As discussed in [4], it is often the case that the natural choice of coordinates for the asymptotic expansion takes the fluctuations outside the de Donder gauge. Indeed, we will find here that there is a distinguished coordinate choice which is outside de Donder gauge. This issue may be dealt with using gauge invariant combinations of the fluctuations; these were derived up to quadratic order in the fluctuations in [4]. For the purposes of this paper we will need only certain combinations which are gauge invariant at linear order, namely:

$$\begin{aligned} \hat{\pi}^{kI_1} &= \pi^{kI_1} - \Lambda^{I_1} \phi^{kI_1}_{(s)} \\ \hat{B}^{kI_5}_{(v)\mu} &= B^{kI_5}_{(v)\mu} - \frac{1}{2} D_\mu \phi^{kI_5}_{(v)} \\ \hat{b}^{kI_1} &= b^{kI_1}_{(s)} - \frac{1}{2} \phi^{kI_1}_{(s)} \\ \hat{b}^{kI_5}_\mu &= b^{kI_5}_\mu - \frac{1}{2(\Lambda^{I_5} - 4)} D_\mu \phi^{kI_5}_{(v)}. \end{aligned} \tag{2.9}$$

Note also that $h^0_{\mu\nu}$ is a deformation of the background metric and it indeed transforms as a metric.

2.2 The spectrum

In this subsection we review the relevant parts of the spectrum of fluctuations about $AdS_5 \times S^5$ computed in [31]. As discussed in detail in [4], one can relax the de Donder gauge fixing condition used in [31] by replacing all fields by the gauge invariant (hatted) versions given in the previous section.

The scalars relevant here satisfy the following linearized equations

$$\begin{aligned} \Box \hat{s}^{kI_1} &= k(k-4)\hat{s}^{kI_1}, \quad k \geq 2, \\ \Box \hat{t}^{kI_1} &= (k+4)(k+8)\hat{t}^{kI_1}, \quad k \geq 0, \end{aligned} \tag{2.10}$$

where we introduce the combinations

$$\hat{s}^{kI_1} = \frac{1}{20(k+2)}(\hat{\pi}^{kI_1} - 10(k+4)\hat{b}^{kI_1}), \qquad \hat{t}^{kI_1} = \frac{1}{20(k+2)}(\hat{\pi}^{kI_1} + 10k\hat{b}^{kI_1}), \tag{2.11}$$

with inverse relations $\hat{b}^{kI_1} = -\hat{s}^{kI_1} + \hat{t}^{kI_1}$, $\hat{\pi}^{kI_1} = 10k\hat{s}^{kI_1} + 10(k+4)\hat{t}^{kI_1}$. The s^I fields are dual to scalar chiral primary operators.

The relevant vector combinations are

$$\begin{aligned} a_\mu^{kI_5} &= (\hat{B}_{(v)\mu}^{kI_5} - 4(k+3)\hat{b}_\mu^{kI_5}); \\ c_\mu^{kI_5} &= (\hat{B}_{(v)\mu}^{kI_5} + 4(k+1)\hat{b}_\mu^{kI_5}), \end{aligned} \tag{2.12}$$

with the corresponding masses being

$$m^2(a^k) = (k^2 - 1); \quad m^2(c^k) = (k+3)(k+5). \tag{2.13}$$

Thus the $k = 1$ modes of a_μ are massless and are dual to the R symmetry currents.

The combination of $10d$ fields that satisfies the $5d$ linearized Einstein equation is

$$\tilde{h}^0_{\mu\nu} = (h^0_{\mu\nu} + \frac{1}{3}g^o_{\mu\nu}\pi^0); \tag{2.14}$$

the shift by π^0 follows from the Weyl transformation required to bring the $5d$ action into the Einstein frame.

2.3 Kaluza-Klein reduction

The non-linear Kaluza-Klein reduction maps for the scalar fields s^{kI} were computed in [4]. For $k = 2, 3$ only the linear term in the reduction formula is needed, namely

$$S^{kI} = w(s^k)s^{kI}; \quad w(s^k) = \sqrt{\frac{8k(k-1)(k+2)z(k)}{(k+1)}}, \tag{2.15}$$

where five-dimensional fields are denoted with capital letters and $z(k)$ is the spherical harmonic normalization (A.2). For $k = 4$ the quadratic corrections to the reduction formula are also needed, thus

$$S^{4I} = w(s^4)(s^{4I} + J_{IJK}s^{2J}s^{2K} + L_{IJK}D_\mu s^{2J}D^\mu s^{2K}), \tag{2.16}$$

where

$$J_{IJK} = -\frac{2^7\sqrt{5}}{\sqrt{3}\pi^3}a_{IJK}, \quad L_{IJK} = -\frac{40\sqrt{5}}{3\sqrt{3}\pi^3}a_{IJK} \tag{2.17}$$

with a_{IJK} the triple overlap between scalar harmonics (A.3).

The reduction formula for the metric was also determined in [4] to be

$$\begin{aligned} G_{\mu\nu} &= g^o_{\mu\nu} + \tilde{h}^0_{\mu\nu} + L_{\mu\nu}; \qquad (2.18)\\ L_{\mu\nu} &= -\frac{1}{12}\left(\frac{2}{9}D_\mu D^\rho \hat{s}^{2I} D_\nu D_\rho \hat{s}^{2I} - \frac{10}{3}\hat{s}^{2I} D_\mu D_\nu \hat{s}^{2I}\right.\\ &\qquad \left. +(\frac{10}{9}(D\hat{s}^{2I})^2 - \frac{32}{9}(\hat{s}^{2I})^2) g^o_{\mu\nu}\right), \end{aligned}$$

where $G_{\mu\nu}$ is the $5d$ metric. Again the only relevant quadratic corrections are those involving scalars.

The reduction of gauge fields was not discussed in [4] but can be determined from the results of [32] for the quadratic action. That is, the action for the vector fields $a^{kI_5}_\mu$ is given by

$$S = \frac{N^2}{2\pi^2}\int d^5x\sqrt{-G}\left(w(a^k)^2(-\frac{1}{16}F(a^{kI_5})^2 - \frac{1}{8}(k^2-1)(a^{kI_5})^2\right), \qquad (2.19)$$

with $F_{\mu\nu}(a) = \partial_\mu a_\nu - \partial_\nu a_\mu$ and

$$w(a^k) = \sqrt{2\frac{(k+1)z(k)}{(k+2)}}. \qquad (2.20)$$

Here $z(k)$ is again the harmonic normalization, defined in (A.2). Then the linear reduction formula is given by

$$A^{kI_5}_\mu = w(a^k)a^{kI_5}_\mu \qquad (2.21)$$

where the action for the $5d$ fields $A^{kI_5}_\mu$ is

$$S = \frac{N^2}{2\pi^2}\int d^5x\sqrt{-G}\left(-\frac{1}{16}F(A^{kI_5})^2 - \frac{1}{8}(k^2-1)(A^{kI_5})^2\right), \qquad (2.22)$$

Non-linear corrections to this reduction formula will not be needed in what follows since they will not affect the vevs of the R symmetry currents. The normalization of the $5d$ gauge fields is such that the corresponding R symmetry currents have the standard normalization, that is, their two point functions are given by [33]

$$\langle J^a_i(x_1)J^b_j(x_2)\rangle = \frac{N^2}{2(2\pi)^4}\delta^{ab}(\Box\delta_{ij} - \partial_i\partial_j)\frac{1}{(x_1-x_2)^4}, \qquad (2.23)$$

where $4d$ coordinates are labelled by x^i and (a,b) label the SO(6) indices.

2.4 Holographic 1-point functions

The final step is to use the method of holographic renormalization to extract the vevs from the asymptotics of the $5d$ fields. This is by now a standard procedure except that here one needs to include additional terms to accommodate extremal couplings (see section 5.4 of [4]). The relation between field asymptotics and vevs is most transparent in Hamiltonian variables where the radius plays the role of time. The 1-point functions are then related to the radial canonical momenta of the bulk fields, which are expressed as (non-linear) functions of the field asymptotics [28, 29].

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Let us now summarize the expressions for the holographic 1-point functions. Consider first the metric and scalar fields. The near-boundary expansion of the bulk metric $G_{\mu\nu}$ and scalar fields Φ^k, where k is the dimension of the dual operator, take the form

$$\begin{aligned} ds_5^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2}\left(G_{(0)ij}(x) + z^2 G_{(2)ij}(x) + z^4(G_{(4)ij}(x) + \log z^2 h_{(4)ij}(x))\right) dx^i dx^j; \\ \Phi^2(x,z) &= z^2\left(\log z^2 \Phi^2_{(0)}(x) + \tilde{\Phi}^2_{(0)}(x) + \cdots\right); \\ \Phi^k(x,z) &= z^{(4-k)}\Phi^k_{(0)}(x) + \cdots + z^k \Phi^k_{(2k-4)}(x) + \cdots, \qquad k > 2. \end{aligned} \tag{2.24}$$

In these expressions the boundary fields $G_{(0)ij}, \Phi^2_{(0)}, \Phi^k_{(0)}$ parametrize the Dirichlet boundary conditions and are also the field theory sources for the QFT stress energy tensor and operators of dimension 2 and k, respectively. The near-boundary analysis determines all coefficients in these expansions except the ones corresponding to the normalizable modes, namely $G_{(4)ij}, \tilde{\Phi}^2_{(0)}, \Phi^k_{(2k-4)}$.

Consider first the scalar operators $\mathcal{O}_{S^{2I}}$ and $\mathcal{O}_{S^{3I}}$, where I labels their degeneracy. We will be interested in SO(4) singlet operators which can be labeled by their SO(2) charge m, but we will express the holographic relations in a more generally applicable way. For these operators the holographic relations are [4]:

$$\langle \mathcal{O}_{S^{kI}} \rangle = \frac{N^2}{2\pi^2}(\pi^{kI}_{(k)}), \tag{2.25}$$

where $\pi^{km}_{(k)}$ indicates the part of the canonical momentum of the field S^{kI} that scales with weight k. The relevant part of the canonical momenta can be expressed in terms of the asymptotic expansion of the $5d$ fields as follows

$$\pi^{kI}_{(2k-4)} = (2k-4)[S^{kI}]_k \tag{2.26}$$

where the notation $[A]_k$ indicates the coefficient of the z^k term in A and z is the Fefferman-Graham radial coordinate. The relation (2.26) holds for $k \neq 2$; when $k = 2$ one should replaces the factor $(2k-4)$ by 2.

As discussed in some detail in [4] the vevs of the scalar operators of dimension four also involve quadratic terms; these are necessary to accommodate extremal couplings. Thus the vevs in this case are

$$\langle \mathcal{O}_{S^{4I}} \rangle = \frac{N^2}{2\pi^2}\left(\pi^{4I}_{(4)} + 15\sqrt{3} a_{IJK} \pi^{2J}_{(2)} \pi^{2K}_{(2)}\right) \tag{2.27}$$

It is useful to express these vevs directly in terms of the coefficients that appear in the $10d$ solution. Using the results reviewed above one obtains

$$\begin{aligned} \langle \mathcal{O}_{S^{2I}} \rangle &= \frac{N^2}{2\pi^2}\frac{2\sqrt{8}}{3}[s^{2I}]_2; \quad \langle \mathcal{O}_{S^{3I}} \rangle = \frac{N^2}{2\pi^2}\sqrt{3}[s^{3I}]_3; \\ \langle \mathcal{O}_{S^{4I}} \rangle &= \frac{N^2}{2\pi^2}\frac{4\sqrt{3}}{5}[2s^{4I} + \frac{37}{9z(4)} a_{IJK} s^{2J} s^{2K} - \frac{7}{9z(4)} a_{IJK}(D_\mu s^{2J})(D^\mu s^{2K})]_4, \end{aligned} \tag{2.28}$$

where $z(k)$ is the normalization of the degree k spherical harmonics, defined in (A.2). The expression for $\langle \mathcal{O}_{S^{4I}} \rangle$ can be further simplified for solutions in which s^2 has vev (rather than source) behavior, such as those under consideration in this paper. In such cases, the asymptotics of (2.24) imply that

$$[(D_\mu s^{2J} D^\mu s^{2K}]_4 = [z^2 \partial_z s^{2J} \partial_z s^{2K}]_4 = 4[s^{2J} s^{2K}]_4 \tag{2.29}$$

and thus we obtain

$$\langle \mathcal{O}_{S^{4I}} \rangle = \frac{N^2}{2\pi^2} \frac{4\sqrt{3}}{5} \left[2s^{4I} + \frac{1}{z(4)} a_{IJK} s^{2J} s^{2K} \right]_4 . \tag{2.30}$$

Next consider the stress energy tensor; its vev can be obtained by analyzing the coupled system of the metric and the scalar fields S^{2I}. (The other $5d$ fields fall off too fast to contribute to the stress energy tensor.) The part of the $5d$ action involving the metric and one S^2 field is same as the sector of gauged supergravity analyzed in [26, 27], where S^2 was called Φ. The result for the stress energy tensor can thus be carried over from these works, with $S^2 \to S^{2I}$. Thus one gets

$$\begin{aligned} \langle T_{ij} \rangle = \frac{N^2}{2\pi^2} \bigg(& G_{(4)ij} + \frac{1}{3} (\tilde{S}^{2I}_{(0)} \tilde{S}^{2I}_{(0)}) G_{(0)ij} + \frac{1}{8} [\mathrm{Tr} G^2_{(2)} - (\mathrm{Tr} G_{(2)})^2] G_{(0)ij} \\ & - -\frac{1}{2} (G^2_{(2)})_{ij} + \frac{1}{4} G_{(2)ij} \mathrm{Tr} G_{(2)} + \frac{3}{2} h_{(4)ij} \bigg) , \end{aligned} \tag{2.31}$$

where the summation over I is implicit. Again it is useful to rewrite this expression in terms of ten-dimensional fields. First note that the quadratic terms in the reduction formula (2.18) can be written as

$$\begin{aligned} L_{zz} &= \frac{20}{27} z^2 \tilde{s}^{2I}_{(0)} \tilde{s}^{2I}_{(0)} + \cdots ; \qquad L_{zi} = -\frac{z^3}{2} \tilde{s}^{2I}_{(0)} \partial_i \tilde{s}^{2I}_{(0)} + \cdots ; \\ L_{ij} &= -\frac{19}{27} z^2 \tilde{s}^{2I}_{(0)} \tilde{s}^{2I}_{(0)} G_{(0)ij} + \cdots . \end{aligned} \tag{2.32}$$

where $\tilde{s}^{2I}_{(0)}$ is the normalizable mode of s^{2I}, see (2.24). Now the uplifted metric is by definition

$$g^o_{\mu\nu} + h^o_{\mu\nu} = G_{\mu\nu} - L_{\mu\nu}. \tag{2.33}$$

The formula (2.31) assumes that $G_{\mu\nu}$ is in Fefferman-Graham form, but since $(L_{zz}, L_{zi}) \neq 0$ this implies that the uplifted $10d$ metric is not in Fefferman-Graham form. We can however easily rewrite (2.31) in terms of $10d$ metric which is in Fefferman-Graham form by applying a coordinate transformation. This gives the final formula for the vev of the stress energy tensor

$$\begin{aligned} \langle T_{ij} \rangle = \frac{N^2}{2\pi^2} \bigg(& g_{(4)ij} - \frac{2}{3} (\tilde{s}^{2I}_{(0)} \tilde{s}^{2I}_{(0)}) g_{(0)ij} \\ & + \frac{1}{8} [\mathrm{Tr} g^2_{(2)} - (\mathrm{Tr} g_{(2)})^2] g_{(0)ij} - \frac{1}{2} (g^2_{(2)})_{ij} + \frac{1}{4} g_{(2)ij} \mathrm{Tr} g_{(2)} + \frac{3}{2} h_{(4)ij} \bigg) , \end{aligned} \tag{2.34}$$

where $g_{(k)ij}$ are the coefficients in the Fefferman-Graham expansion of the $10d$ metric.

Let us finally consider the R symmetry currents; from the results of [27] their vevs are

$$\langle J^a_i \rangle = -\frac{N^2}{8\pi^2} \tilde{A}^{1a}_i \equiv -\frac{\sqrt{2} N^2}{24\pi^2} \tilde{a}^{1a}_i \tag{2.35}$$

where the gauge field is in radial axial gauge, $A_z = 0$, and the asymptotics of the five-dimensional gauge field (with the source term set to zero) are

$$A_i^{1a} = (z^2 \tilde{A}_i^{1a} + \cdots). \tag{2.36}$$

(where the indices kI_5 in (2.21) are here $k = 1$, $I_5 = a$.) It is again useful to rewrite the R symmetry current in terms of ten-dimensional fields to give the second equality in (2.35).

Before leaving this section, let us comment on the wider applicability of the highlighted expressions for the holographic vevs, (2.28), (2.34) and (2.35). In this paper we will analyse in detail the LLM bubbling solutions, which preserve an $R\times\mathrm{SO}(4)\times\mathrm{SO}(4)$ symmetry group, and are associated with 1/2 BPS states of $\mathcal{N} = 4$ SYM on $R \times S^3$.

However 1/4 and 1/8 BPS states on $R \times S^3$ which are built from operators involving only the six scalars of $\mathcal{N} = 4$ also induce vevs only for the R-currents, the stress energy tensor and the scalar chiral primaries. Thus the expressions for the holographic vevs given here can be used to extract such data from putative dual geometries, of the type constructed in [34]. It would be straightforward to derive corresponding expressions for more general asymptotically $AdS_5 \times S^5$ solutions, which involve more supergravity fields, such as the Janus solutions recently derived in [35]. Note in particular that the expression given here for the stress energy tensor (2.34) provides a rigorous way to extract the mass (including the Casimir term) from the ten-dimensional solution.

3. Bubbling solutions

The LLM bubbling solutions are[4]

$$\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx_i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2; \\
h^{-2} &= 2y\cosh(G); \qquad z = \frac{1}{2}\tanh G; \\
y\partial_y V_i &= \epsilon_{ij}\partial_j z; \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z; \\
F_5 &= F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3; \\
F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B}; \\
\tilde{F} &= d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\tilde{B}; \\
B_t &= -\frac{1}{4}y^2 e^{2G}; \qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G}; \\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right); \qquad d\tilde{B} = -\frac{1}{4}y^3 *_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right),
\end{aligned} \tag{3.1}$$

where $i = 1, 2$ and $*_3$ is the Hodge dual on the R^3 parameterized by (y, x_1, x_2). The solutions are characterized by a harmonic function on six dimensions, with sources on an R^2. That is,

$$\frac{z(x_1, x_2, y)}{y^2} = \frac{1}{\pi}\int_{R^2} \frac{z(x_1', x_2', 0)dx_1' dx_2'}{((x - x')^2 + y^2)^2}, \tag{3.2}$$

[4]Note that we use the notation z with two completely different meanings; as the function z defined in (3.1) and also as the Fefferman-Graham radial coordinate, (2.24). The meaning of z should be clear from the context.

where regularity of the solution requires that $z(x_1', x_2', 0)$ takes the values $\pm\frac{1}{2}$. The two-dimensional vector V_i can be written as

$$V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi} \int_{R^2} \frac{z(x_1', x_2', 0)(x_j - x_j')dx_1' dx_2'}{((x - x')^2 + y^2)^2}. \tag{3.3}$$

In polar coordinates on R^2 this can be written as

$$V_{\tilde{\phi}} = -\frac{r}{\pi} \int_{R^2} \frac{z(r', \tilde{\phi}', 0)(r - r' \cos(\tilde{\phi} - \tilde{\phi}'))r' dr' d\tilde{\phi}'}{((x - x')^2 + y^2)^2}; \tag{3.4}$$

$$V_r = \frac{1}{\pi} \int_{R^2} \frac{z(r', \tilde{\phi}', 0) \sin(\tilde{\phi} - \tilde{\phi}')(r')^2 dr' d\tilde{\phi}'}{((x - x')^2 + y^2)^2}. \tag{3.5}$$

3.1 $AdS_5 \times S^5$ solution

The $AdS_5 \times S^5$ solution is obtained by taking sources for z on a disk of radius r_0. Then

$$z^o = -\frac{1}{2} \left(\frac{(r^2 + y^2 - r_0^2)}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} \right) \tag{3.6}$$

where r is a polar coordinate on R^2 such that $x_1 = r \cos\tilde{\phi}$ and $x_2 = r \sin\tilde{\phi}$. Introducing the following coordinate change on the R^2 parameterized by y, x_1, x_2

$$y \equiv \tilde{R} \cos\tilde{\theta} = R \cos\theta; \quad r \equiv \tilde{R} \sin\tilde{\theta} = \sqrt{R^2 + r_0^2} \sin\theta; \quad \tilde{\phi} = \phi - t, \tag{3.7}$$

gives

$$z^o = -\frac{1}{2} + \frac{r_0^2 \cos^2\theta}{(R^2 + r_0^2 \cos^2\theta)}. \tag{3.8}$$

The coordinate shift (3.7) changes the flat metric on R^3

$$ds_3^2 = d\tilde{R}^2 + \tilde{R}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2) \tag{3.9}$$

to the following metric:

$$ds_3^2 = (R^2 + r_0^2 \cos^2\theta) \left(\frac{dR^2}{R^2 + r_0^2} + d\theta^2 \right) + (R^2 + r_0^2) \sin^2\theta (d\phi - dt)^2. \tag{3.10}$$

The other functions in the metric take the values

$$(h^{-2})^o = r_0^{-1}(R^2 + r_0^2 \cos^2\theta); \qquad (ye^G)^o = r_0 \cos^2\theta; \tag{3.11}$$

$$V^o = -\frac{r_0^2 \sin^2\theta}{(R^2 + r_0^2 \cos^2\theta)}(d\phi - dt); \qquad (ye^{-G})^o = \frac{R^2}{r_0}.$$

The superscript A^o denotes that these are the background $AdS_5 \times S^5$ functions, about which we will expand. Substituting these values into the metric gives

$$ds^2 = r_0 \left(-(\hat{R}^2 + 1)dt^2 + \frac{d\hat{R}^2}{(\hat{R}^2 + 1)} + \hat{R}^2 d\tilde{\Omega}_3^2 + (d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2) \right), \tag{3.12}$$

where $\hat{R} = R/r_0$. This is indeed the metric on $AdS_5 \times S^5$ with global coordinates on AdS_5 and curvature radius $\sqrt{r_0}$; henceforth r_0 will be set to one.

The five form field strength can be obtained in the following way. The two form $d\hat{B}$ is given by

$$
\begin{aligned}
(d\hat{B})_{\theta\phi} &= -(d\hat{B})_{\theta t} = -\frac{1}{4}R^3(R^2+1)\cos^3\theta\sin\theta\partial_R\Phi; \\
(d\hat{B})_{R\phi} &= -(d\hat{B})_{Rt} = \frac{1}{4}R^3\cos^3\theta\sin\theta\partial_\theta\Phi; \\
(d\hat{B})_{R\theta} &= -\frac{1}{4}R^3\cos^3\theta\frac{(R^2+\cos^2\theta)}{\sin\theta(R^2+1)}\partial_\phi\Phi,
\end{aligned}
\tag{3.13}
$$

where

$$
\Phi = y^{-2}(z+\frac{1}{2}), \tag{3.14}
$$

and for $AdS_5 \times S^5$

$$
\Phi^o = \frac{1}{R^2(R^2+\cos^2\theta)}. \tag{3.15}
$$

The two form $d\tilde{B}$ is similarly given by

$$
\begin{aligned}
(d\tilde{B})_{\theta\phi} &= -(d\tilde{B})_{\theta t} = -\frac{1}{2}\cos\theta\sin\theta(R^2+1) - \frac{1}{4}R^3(R^2+1)\cos^3\theta\sin\theta\partial_R\Phi; \\
(d\tilde{B})_{R\phi} &= -(d\tilde{B})_{Rt} = -\frac{1}{2}R\sin^2\theta + \frac{1}{4}R^3\cos^3\theta\sin\theta\partial_\theta\Phi; \\
(d\tilde{B})_{R\theta} &= -\frac{1}{4}R^3\cos^3\theta\frac{(R^2+\cos^2\theta)}{\sin\theta(R^2+1)}\partial_\phi\Phi,
\end{aligned}
\tag{3.16}
$$

Substituting into the expression for the five form then gives the following expression for $AdS_5 \times S^5$:

$$
\tilde{F}^o_{tR} = R^3; \quad F^o_{\theta\phi} = \cos^3\theta\sin\theta, \tag{3.17}
$$

as expected.

3.2 Asymptotic expansion

Now let us consider more general solutions which are asymptotic to $AdS_5 \times S^5$. The field theory data will be extracted from their asymptotic expansions around the $AdS_5 \times S^5$ boundary. This expansion can be economically expressed as follows. Let the solution be expressed in terms of the harmonic function $\Phi(x_1, x_2, y)$ with

$$
\Phi = \Phi^o + \Delta\Phi, \tag{3.18}
$$

where Φ^o is the harmonic function of the $AdS_5 \times S^5$ background about which we perturb. $\Delta\Phi$ can be expressed as

$$
\Delta\Phi(x_1, x_2, y) = \frac{1}{\pi}\int_{R^2}\frac{\Delta z(x_1', x_2', 0)dx_1'dx_2'}{((x-x')^2+y^2)^2}, \tag{3.19}
$$

where $\Delta z(x_1', x_2', 0) = (z(x_1', x_2', 0) - z^o(x_1', x_2', 0))$. Now note that Φ (and hence $\Delta\Phi$) is a scalar harmonic function on R^6 which preserves SO(4) rotational symmetry. The asymptotics can thus be expressed as

$$\Delta\Phi(\tilde{R}, \tilde{\theta}, \tilde{\phi}) = \sum_{k,m} (\Delta\Phi)_{km} \frac{Y_k^m(\tilde{\theta}, \tilde{\phi})}{\tilde{R}^{k+4}}, \tag{3.20}$$

where $Y_k^m(\tilde{\theta}, \tilde{\phi})$ are normalized SO(4) singlet spherical harmonics of degree k with m labeling their SO(2) charge; the properties of such harmonics are discussed in appendix A. By the addition theorem the coefficients in this expansion are given by [2]

$$(\Delta\Phi)_{km} = 2^k (k+1) \pi^{-1} \int_{R^2} \Delta z(x_1', x_2', 0) (C^m_{i_1 \cdots i_k} x^{i_1'} \cdots x^{i_k'}) dx_1' dx_2', \tag{3.21}$$

where $C^m_{i_1 \cdots i_k}$ are SO(4) invariant symmetric traceless tensors on R^6 of rank k which are in one to one correspondence with the SO(4) singlet spherical harmonics. In particular

$$(\Delta\Phi)_{20} = 4\sqrt{3}\pi^{-1} \int_{R^2} \Delta z(x_1', x_2', 0)(r')^3 dr' d\phi', \tag{3.22}$$

where the explicit representation of the SO(4) $\times$ SO(2) singlet tensor is used.

Note that the expansion (3.20) begins at $k = 2$. There is no $k = 0$ term, since the leading asymptotics are those of $AdS_5 \times S^5$ and $k = 1$ terms are unphysical since they can always be removed by choosing the origin of the coordinate system to be at the centre of mass. The centre of mass conditions imply that

$$\int_{R^2} z(r', \tilde{\phi}', 0) r' e^{\pm i\phi'} (r' dr' d\tilde{\phi}') = \int_{R^2} \Delta z(r', \tilde{\phi}', 0) r' e^{\pm i\phi'} (r' dr' d\tilde{\phi}') = 0. \tag{3.23}$$

The harmonic function (3.20) is expressed in the usual coordinates on R^6, but to perturb relative to $AdS_5 \times S^5$ the function needs to expressed in terms of the coordinates (R, θ, ϕ). However, for $(R, \tilde{R}) \gg 1$, this change in coordinates changes the form of (3.20) only at $k \geq 4$, that is,

$$\begin{aligned} \Delta\Phi(R, \theta, \phi, t) = {} & (\Delta\Phi)_{2m} \frac{Y_2^m(\theta, \phi - t)}{R^6} + (\Delta\Phi)_{3m} \frac{Y_3^m(\theta, \phi - t)}{R^7} + \\ & \frac{1}{R^8} ((\Delta\Phi)_{4m} Y_4^m(\theta, \phi - t) + (\Delta\Phi)_{2m} f^m(\theta, \phi - t)) + \cdots \end{aligned} \tag{3.24}$$

where the functions $f^m(\theta, \phi - t)$ are given by

$$\begin{aligned} f^0(\theta, \phi - t) &= f_0^0 Y_0 + f_2^0 Y_2^0(\theta) + f_4^0 Y_4^0(\theta); \\ f^{\pm 2}(\theta, \phi - t) &= f_2^{\pm 2} Y_2^{\pm 2}(\theta, \phi - t) + f_4^{\pm 2} Y_4^{\pm 2}(\theta, \phi - t). \end{aligned} \tag{3.25}$$

Note that $Y_0 = 1$. These coefficients are obtained by expanding $\Delta\Phi(\tilde{R}, \tilde{\theta}, \tilde{\phi})$ using

$$\frac{1}{\tilde{R}^2} = \frac{1}{R^2}\left(1 - \frac{\sin^2\theta}{R^2} + \cdots\right); \quad \sin\tilde{\theta} = \sin\theta\left(1 + \frac{\cos^2\theta}{2R^2} + \cdots\right), \tag{3.26}$$

and then projecting back onto the basis of spherical harmonics. In what follows we will need only the following coefficients explicitly

$$f_0^0 = 0; \qquad f_4^0 = -\frac{4\sqrt{3}}{\sqrt{5}}; \qquad f_4^{\pm 2} = -\frac{8}{\sqrt{5}}. \tag{3.27}$$

The functions appearing in the metric can be expressed in terms of $\Delta\Phi$ as follows

$$\begin{aligned} ye^G &= \cos^2\theta(1+R^2(R^2+\cos^2\theta)\Delta\Phi)^{\frac{1}{2}}(1-\cos^2\theta(R^2+\cos^2\theta)\Delta\Phi)^{-\frac{1}{2}} \\ &\equiv \cos^2\theta(1+\alpha); \\ ye^{-G} &= R^2(1+R^2(R^2+\cos^2\theta)\Delta\Phi)^{-\frac{1}{2}}(1-\cos^2\theta(R^2+\cos^2\theta)\Delta\Phi)^{\frac{1}{2}} \\ &\equiv R^2(1+\beta); \\ h^{-2}(R^2+\cos^2\theta)^{-1} &= (1+(R^4-\cos^2\theta)\Delta\Phi - R^2\cos^2\theta(R^2+\cos^2\theta)^2\Delta\Phi^2)^{-\frac{1}{2}} \\ &\equiv (1+\gamma); \\ h^2(R^2+\cos^2\theta) &= (1+(R^4-\cos^2\theta)\Delta\Phi - R^2\cos^2\theta(R^2+\cos^2\theta)^2\Delta\Phi^2)^{\frac{1}{2}} \\ &\equiv (1+\delta). \end{aligned}$$

Note that the leading order terms in $(\alpha, \beta\,\gamma, \delta)$ are of order $1/R^2$; to extract vevs of operators of dimension four and less it will be sufficient to expand these quantities up to order $1/R^4$. Then

$$\begin{aligned} \alpha &= \left(\frac{1}{2}R^4\Delta\Phi + R^2\cos^2\theta\Delta\Phi - \frac{1}{8}R^8(\Delta\Phi)^2 + \cdots\right); \qquad (3.28) \\ \beta &= \left(-\frac{1}{2}R^4\Delta\Phi - R^2\cos^2\theta\Delta\Phi + \frac{3}{8}R^8(\Delta\Phi)^2 + \cdots\right); \\ \gamma &= \left(-\frac{1}{2}R^4\Delta\Phi + \frac{3}{8}R^8(\Delta\Phi)^2 + \cdots\right); \\ \delta &= \left(\frac{1}{2}R^4\Delta\Phi - \frac{1}{8}R^8(\Delta\Phi)^2 + \cdots\right). \end{aligned}$$

Now consider the vector V_i. The centre of mass conditions imply that $(\Delta V) \equiv (V - V^o)$ is given by

$$\Delta V_{\tilde{\phi}} = \frac{v_{\tilde{\phi}}(\tilde{\theta},\tilde{\phi})}{\tilde{R}^4} + \cdots; \qquad \Delta V_r = \frac{v_r(\tilde{\theta},\tilde{\phi})}{\tilde{R}^5} + \cdots, \tag{3.29}$$

with

$$\begin{aligned} v_{\tilde{\phi}}(\tilde{\theta},\tilde{\phi}) &= -R^6\sin^2\tilde{\theta}\Delta\Phi + \frac{1}{6}\left(Y^{2,2}(\Delta\Phi)_{22} + Y^{2,-2}(\Delta\Phi)_{2(-2)} + (\Delta\Phi)_{20}(2Y^{2,0} + \frac{1}{\sqrt{3}}Y^0)\right) \\ v_r(\tilde{\theta},\tilde{\phi}) &= -\frac{i}{6}\sin\tilde{\theta}\left(e^{2i\tilde{\phi}}\Delta\Phi_{22} - e^{-2i\tilde{\phi}}\Delta\Phi_{2(-2)}\right). \qquad (3.30) \end{aligned}$$

Thus

$$\Delta V_\phi = \Delta V_t = \frac{v_{\tilde{\phi}}}{R^4} + \cdots; \tag{3.31}$$

$$\Delta V_R = \sin\theta\frac{v_r}{R^5} + \cdots; \qquad \Delta V_\theta = \cos\theta\frac{v_r}{R^4} + \cdots \tag{3.32}$$

3.3 Expansion of metric and five form

The asymptotic expansion of the metric is given by

$$
\begin{aligned}
ds^2 = & -dt^2\Big((R^2+1)(1+\gamma)+\sin^2\theta(\gamma-\delta)-2\frac{v_{\tilde\phi}}{R^2}\Big)+R^2(1+\beta)d\Omega_3^2+(1+\delta)\frac{dR^2}{R^2+1} \\
& -2dtdR\frac{v_r\sin\theta}{R^3}-2dtd\theta\frac{v_r\cos\theta}{R^2}+2dtd\phi\Big((\gamma-\delta)\sin^2\theta-\frac{v_{\tilde\phi}}{R^2}\Big) \\
& +(1+\delta)d\theta^2+\cos^2\theta(1+\alpha)d\Omega_3^2+2\sin^2\theta d\phi\Big(\frac{d\theta\cos\theta}{R^4}\Big)v_r \\
& \sin^2\theta d\phi^2\Big(1+\delta+\frac{\sin^2\theta}{R^2}(\delta-\gamma)+2\frac{v_{\tilde\phi}}{R^4}\Big)+\cdots,
\end{aligned}
\tag{3.33}
$$

where the terms retained are sufficient to extract vevs of operators of dimension four and less. That is, to extract the vevs of the scalar operators with dimension less than or equal to four one needs $(\pi,\phi_{(s)})$ to order $1/R^4$. To extract the vev of the R symmetry current one will need $B_{(v)\mu}$ with $\mu\neq R$ to order $1/R^2$. For the vev of the stress energy tensor one needs $(\pi^0,\tilde h^0_{\mu\nu})$ up to order $1/R^4$.

In actually extracting these fields there is considerable simplification relative to the discussions of [4]. Consider first the perturbations tangent to the sphere and note that

$$
h_{ab}dx^a dx^b=(1+\alpha)(d\theta^2+\cos^2\theta d\Omega_3^2+\sin^2\theta d\phi^2)+\mathcal{O}\Big(\frac{1}{R^4}\Big). \tag{3.34}
$$

This implies that to order $1/R^4$ the metric perturbation along the sphere is in de Donder gauge, with only the trace π non-vanishing. Compared to [4] where the fields $\phi_{(s)}$ (zero in de Donder gauge) were already excited at order $1/R^2$ there is simplification. Gauge invariant combinations of fluctuations are needed first at order $1/R^4$ (or correspondingly when computing vevs of operators of dimension four) and moreover only linearly gauge invariant quantities will be needed at this order. One would only need to use gauge invariant quantities at non-linear order for computing vevs of operators with dimension greater than four. Thus one can immediately extract

$$
\hat\pi^{km}=\pi^{km}=\frac{5}{2R^k}(\Delta\Phi)_{km}e^{-imt}+\cdots;\qquad k=2,3. \tag{3.35}
$$

To obtain $(\pi^0,\pi^4,\phi^4_{(s)})$ we split the sphere perturbation into its trace and traceless parts:

$$
\begin{aligned}
\pi &= \Big(5\delta+R^2(3-2\sin^2\theta)\Delta\Phi+\frac{2v_{\tilde\phi}}{R^4}+\cdots\Big); \qquad (3.36)\\
h_{(\theta\theta)} &= \Big(-\frac{1}{5}R^2\Delta\Phi(3-2\sin^2\theta)-\frac{2v_{\tilde\phi}}{5R^4}+\cdots\Big);\\
h_{(\phi\phi)} &= \sin^2\theta\left(\frac{1}{5}R^2\Delta\Phi(7\sin^2\theta-3)+\frac{8v_{\tilde\phi}}{5R^4}+\cdots\right);\\
h_{(\chi^\alpha\chi^\beta)} &= \cos^2\theta\hat g_{\chi^\alpha\chi^\beta}\left(\frac{1}{5}R^2\Delta\Phi(2-3\sin^2\theta)-\frac{2v_{\tilde\phi}}{5R^4}+\cdots\right),
\end{aligned}
$$

with $h_{\theta\phi}$ as given in (3.33), χ^α are coordinates on S^3 and $\hat{g}_{\chi^\alpha\chi^\beta}$ is a unit radius metric on S^3. Projecting onto the basis of spherical harmonics gives

$$\begin{aligned}
\pi^{4m} &= e^{-imt}\left(\frac{5}{2R^4}(\Delta\Phi)_{4m} - \frac{5}{8R^4 z_4}a_{mpq}(\Delta\Phi_{2n}\Delta\Phi_{2q}) + 168\phi_{(s)}^{4m} + \cdots\right); \qquad (3.37)\\
\phi_{(s)}^{4\pm2} &= \left(-\frac{\sqrt{2}}{12\sqrt{5}R^4}(\Delta\Phi)_{2\pm2} + \cdots\right);\\
\phi_{(s)}^{40} &= \left(-\frac{\sqrt{3}}{12\sqrt{5}R^4}(\Delta\Phi)_{20} + \cdots\right);\\
\pi^0 &= \left(-\frac{5}{8R^4}(\Delta\Phi_{2n}\Delta\Phi_{2(-n)}) + \cdots\right).
\end{aligned}$$

Thus the gauge invariant combinations are

$$\hat{\pi}^{4m} = e^{-imt}\left(\frac{5}{2R^4}(\Delta\Phi)_{4m} - \frac{5}{8R^4 z_4}a_{mnp}(\Delta\Phi_{2n}\Delta\Phi_{2p}) + 200\phi_{(s)}^{4m} + \cdots\right); \qquad (3.38)$$

Now consider the vector fields. The (non-zero) metric fluctuations $h_{\mu a}$ can be expressed as

$$h_{ta} = \frac{i}{6R^2}D_a\left(Y^{2,2}(\Delta\Phi)_{22} - Y^{2,-2}(\Delta\Phi)_{2(-2)}\right) - \frac{1}{\sqrt{6}R^2}(\Delta\Phi)_{20}Y_a^1. \qquad (3.39)$$

The physical vector fields arise from the projection of the $h_{\mu a}$ terms onto vector harmonics to give $B_{(v)\mu}^{I_5}$. The non-zero projection of $h_{\mu a}$ onto scalar harmonics takes the metric outside de Donder gauge, but the resulting vectors $B_{(s)\mu}^{I_1}$ do not contribute to any gauge invariant quantities computed here. Thus the only relevant vector term is

$$B_{(v)t}^1 = -\frac{1}{\sqrt{6}R^2}(\Delta\Phi)_{20}. \qquad (3.40)$$

Finally let us consider the metric perturbation. The perturbation $\tilde{h}_{\mu\nu}^0$ receives contributions only from the first line in (3.33) since the h_{tR} term does not project onto Y^0; $v_r\sin\theta$ projects only onto $Y^{2\pm2}$. Thus the metric $g_{\mu\nu}^o + \tilde{h}_{\mu\nu}^0$ with $\tilde{h}_{\mu\nu}^0 = h_{\mu\nu}^0 + \frac{1}{3}\pi^0 g_{\mu\nu}^o$ is given by

$$\begin{aligned}
ds^2 &= -dt^2\left(R^2 + 1 - \frac{1}{4\sqrt{3}R^2}(\Delta\Phi)_{20} + \frac{1}{6R^2}(\Delta\Phi_{2n}\Delta\Phi_{2(-n)})\right) \qquad (3.41)\\
&\quad + \frac{dR^2}{(R^2+1)}\left(1 - \frac{1}{3R^4}(\Delta\Phi_{2n}\Delta\Phi_{2(-n)})\right)\\
&\quad + d\tilde{\Omega}_3^2\left(1 + \frac{1}{12\sqrt{3}R^2}(\Delta\Phi)_{20} + \frac{1}{6R^2}(\Delta\Phi_{2n}\Delta\Phi_{2(-n)})\right),
\end{aligned}$$

where summation over $n = (-2, 0, 2)$ is implicit.

Next consider the five form field strength. To compute the vevs we need only the modes $(b_\mu^{I_5}, b_{(s)}^I)$ in the expansion. This means that we need only expand $f_{\mu a} \equiv (F_{\mu\alpha})$ and

$f_{\theta\phi} \equiv (F_{\theta\phi} - F^o_{\theta\phi})$ giving

$$
\begin{aligned}
f_{\theta\phi} &= \sin\theta\cos^3\theta\left(-\frac{1}{4}R^3(R^2+1)\partial_R\Delta\Phi + \frac{\alpha}{R^2}(1-3\sin^2\theta) + \sin\theta\cos\theta\frac{\partial_\theta\alpha}{2R^2}\right. \\
&\qquad\qquad\left. + \frac{v_{\tilde\phi}}{R^4} - \frac{\cos\theta}{4R^4\sin\theta}(\partial_\theta v_{\tilde\phi} - \cos\theta\partial_\phi v_r) + \cdots\right); \\
f_{R\theta} &= \sin\theta\cos^3\theta\left(-\frac{1}{4}R^3\frac{\partial_\phi\Delta\Phi}{\sin^2\theta} + \cdots\right); \qquad (3.42) \\
f_{R\phi} &= \sin\theta\cos^3\theta\left(\frac{1}{4}R^3\partial_\theta\Delta\Phi + \cdots\right); \\
f_{\theta t} &= \sin\theta\cos^3\theta\left(\frac{1}{4}R^5\partial_R(\Delta\Phi) + 2\alpha - \frac{\cos\theta}{2\sin\theta}\partial_\theta\alpha + \cdots\right); \\
f_{\phi t} &= -\frac{1}{2}\cos^4\theta\partial_\phi\alpha + \cdots.
\end{aligned}
$$

From the $f_{\theta t}$ term one gets

$$
f_{\theta t} = \sin\theta\cos^3\theta\left(- \frac{1}{2\sqrt{3}R^2}(\Delta\Phi)_{20} + \cdots\right), \qquad (3.43)
$$

from which one can extract

$$
b^1_t = -\frac{1}{8\sqrt{6}R^2}(\Delta\Phi)_{20}. \qquad (3.44)
$$

Combining this with the vector (3.40) extracted from the metric one finds that

$$
a^1_t = -\frac{\sqrt{3}}{\sqrt{2}R^2}(\Delta\Phi)_{20}; \qquad c^1_t = 0. \qquad (3.45)
$$

This is the anticipated result since the massive vector c^1 should not be excited at this order.

From the $f_{\theta\phi}$ terms one finds

$$
\begin{aligned}
b^{km}_{(s)} &= -\frac{1}{4kR^k}e^{-imt}(\Delta\Phi)_{km}; \qquad k = 2,3 \qquad (3.46) \\
b^{4m}_{(s)} &= \left(-\frac{1}{16R^4}e^{-imt}(\Delta\Phi)_{4m} - \frac{9}{2}\phi^{4m}_{(s)}\right),
\end{aligned}
$$

and therefore the gauge invariant quantities are

$$
\hat{b}^{km}_{(s)} = -\frac{1}{4kR^k}e^{-imt}(\Delta\Phi)_{km} - 5\phi^{4m}_{(s)}; \qquad k = 2,3,4. \qquad (3.47)
$$

Putting together (3.38) and (3.47) gives

$$
\begin{aligned}
\hat{s}^{km} &= e^{-imt}\left(\frac{1}{4kR^k}(\Delta\Phi)_{km} + \left(5\phi^{4m}_{(s)} - \frac{1}{96R^4z_4}a_{mnp}(\Delta\Phi_{2n}\Delta\Phi_{2p})\right)\delta_{k4} + \cdots\right). \quad k = 2,3,4; \\
\hat{t}^{4m} &= -\frac{1}{64R^4z_4}e^{-imt}a_{mnp}(\Delta\Phi_{2n}\Delta\Phi_{2p}). \qquad (3.48)
\end{aligned}
$$

Note that there are no $\phi^{4m}_{(s)}$ terms in the gauge invariant fields $\hat{t}^{4m}$. This is a computational check: using [4] these fields satisfy the field equations

$$(\Box - 96)\hat{t}^{4m} = 96 z_4^{-1} a_{mnp} \hat{s}^{2n} \hat{s}^{2p}, \tag{3.49}$$

and thus at order $1/R^4$ can only receive contributions quadratic in $\Delta\Phi_{2n}$. The $\phi^{4m}_{(s)}$ terms are linear in $\Delta\Phi_{2n}$ and thus cannot contribute to the fields $\hat{t}^{4m}$ at this order.

3.4 Holographic vevs

Given the asymptotic expansions of the relevant fields we can extract the values for the vevs using the formulae from section 2.4. The relation for the R symmetry current vev (2.35) along with (3.45) implies that

$$\langle J_t \rangle = \frac{N^2}{2\pi^2} \frac{1}{4\sqrt{3}} (\Delta\Phi)_{20}. \tag{3.50}$$

To apply the formula (2.34) for the vev of the stress energy tensor one must first bring the metric (3.41) into Fefferman-Graham form, by the coordinate change

$$z = \frac{1}{R}\left(1 - \frac{1}{4R^2} + \frac{19}{128R^4} - \frac{1}{24R^4}(\Delta\Phi)_{2n}(\Delta\Phi)_{2(-n)}\right). \tag{3.51}$$

Then

$$\langle T_{tt} \rangle = \frac{N^2}{2\pi^2}\left(\frac{3}{16} + \frac{1}{4\sqrt{3}}(\Delta\Phi)_{20}\right); \tag{3.52}$$

$$\langle T_{\alpha\beta} \rangle = \frac{N^2}{2\pi^2}\left(\frac{1}{16} + \frac{1}{12\sqrt{3}}(\Delta\Phi)_{20}\right) g_{\alpha\beta}; \tag{3.53}$$

where $g_{\alpha\beta}$ is the metric on the unit radius S^3. Using the explicit form for $(\Delta\Phi)_{20}$ from (3.22) and reinstating factors of a, the inverse radius of the S^3, gives

$$\begin{aligned} \langle J_t \rangle &= \frac{N^2}{2\pi^2} a \int_{R^2} \rho\left(r^2 - \frac{1}{2}a^2\right) r dr d\phi; \qquad (3.54)\\ \langle T_{tt} \rangle &= \frac{N^2}{2\pi^2}\left(\frac{3a^4}{16} + a^2 \int_{R^2} \rho\left(r^2 - \frac{1}{2}a^2\right) r dr d\phi\right) = \langle T_{tt}\rangle_c + a\langle J_t\rangle, \end{aligned}$$

where $\langle T_{tt}\rangle_c$ is the Casimir on $R \times S^3$ and the density function $\rho(r,\phi)$ satisfies

$$\int_{R^2} \rho(r,\phi) r dr d\phi = 1; \qquad \rho^o = \frac{1}{\pi a^2}\theta(a - r). \tag{3.55}$$

We define $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ otherwise. A general distribution is such that $\rho(r,\phi)$ takes the value $1/\pi a^2$ in a region of the plane with area πa^2, and is zero everywhere else. The corresponding mass E and R-charge J are given by integrating these expressions over the S^3, resulting in

$$\begin{aligned} J &= N^2 a \int_{R^2} \rho\left(r^2 - \frac{1}{2}a^2\right) r dr d\phi; \qquad (3.56)\\ E &= N^2\left(\frac{3a^4}{16} + a^2 \int_{R^2} \rho\left(r^2 - \frac{1}{2}a^2\right) r dr d\phi\right) = E_c + aJ. \end{aligned}$$

These quantities have the expected behavior, namely $J = 0$ for AdS with the Casimir energy E_c taking the expected value; the energy and angular momentum tend to zero in the limit of a large S^3 and the BPS bound $(E - E_c) = aJ$ is saturated.

For the scalar operators (2.30) along with (3.48) implies the following result for the vevs:

$$\begin{aligned}
\langle \mathcal{O}_{S^{km}} \rangle &= \frac{N^2}{\pi^2} \frac{(k-2)}{2^{\frac{1}{2}k}(k+1)} \sqrt{\frac{(k-1)}{k}} e^{-iamt} \left((\Delta\Phi)_{km} + 80 R^4 \phi^{4m}_{(s)} \delta_{k4} \right); \\
80 R^4 \phi^{4\pm 2}_{(s)} &= -\frac{4\sqrt{10}}{3} (\Delta\Phi)_{2\pm 2}; \quad 80 R^4 \phi^{40}_{(s)} = -\frac{4\sqrt{5}}{\sqrt{3}} (\Delta\Phi)_{20},
\end{aligned} \tag{3.57}$$

with $(k-2) \to 1$ for $k = 2$ and the scale a, the inverse radius of the S^3, reinstated. For operators with $|m| = k$ (and $k \neq 1$) the vevs are therefore

$$\langle \mathcal{O}_{S^{k\pm k}} \rangle = \frac{N^2}{\sqrt{k}\pi^2} (k-2) \sqrt{k-1} e^{-iakt} \int_{R^2} (r^k \rho) e^{\pm ik\phi} r dr d\phi. \tag{3.58}$$

Recall that there is no $k = 1$ operator in the SU(N) theory; the integral vanishes in this case because of the centre of mass condition (3.23). For the other operators with dimension less than four, one gets

$$\begin{aligned}
\langle \mathcal{O}_{S^{20}} \rangle &= \frac{\sqrt{2} N^2}{\sqrt{3}\pi^2} \int_{R^2} (r^2 \Delta z) r dr d\phi; \\
&= \frac{\sqrt{2} N^2}{\sqrt{3}\pi^2} \left(\int_{R^2} \rho \left(r^2 - \frac{1}{2} a^2 \right) r dr d\phi \right); \\
\langle \mathcal{O}_{S^{3\pm 1}} \rangle &= \frac{N^2}{\pi^2} e^{\mp iat} \int_{R^2} (r^3 \rho) e^{\pm i\phi} r dr d\phi,
\end{aligned} \tag{3.59}$$

where in the second expression for the neutral operator the explicit form of ρ^o is used. For the operators with dimension four, again reinstating the inverse radius of the S^3 one finds

$$\begin{aligned}
\langle \mathcal{O}_{S^{40}} \rangle &= \frac{\sqrt{3} N^2}{\sqrt{5}\pi^2} \int_{R^2} \Delta z (3r^4 - 4a^2 r^2) r dr d\phi; \\
&= \frac{\sqrt{3} N^2}{\sqrt{5}\pi^2} \left(\int_{R^2} \rho (3r^4 - 4a^2 r^2 + a^4) r dr d\phi \right); \\
\langle \mathcal{O}_{S^{4\pm 2}} \rangle &= \frac{4\sqrt{3} N^2}{\sqrt{10}\pi^2} e^{\mp 2iat} \int_{R^2} \rho (r^4 - a^2 r^2) e^{\pm 2i\phi} r dr d\phi.
\end{aligned} \tag{3.60}$$

The general structure of the vevs is thus

$$\langle \mathcal{O}_{S^{km}} \rangle = N^2 e^{-imat} \sum_{l=0}^{\frac{1}{2}(k-|m|)} \alpha_l \int_{R^2} \rho (r^{k-2l} a^{2l}) e^{im\phi} r dr d\phi, \tag{3.61}$$

with certain coefficients α_l. These vevs can also be written in the form

$$\langle \mathcal{O}_{S^{km}} \rangle = \mathcal{M}_k e^{-iamt} (\Delta\Phi)_{km} + \alpha_{km} a^2 \langle \mathcal{O}_{S^{(k-2)m}} \rangle + \beta_{km} a^4 \langle \mathcal{O}_{S^{(k-4)m}} \rangle + \cdots \tag{3.62}$$

where $(\mathcal{M}_k, \alpha_{km})$ are appropriate constants. In the $a \to 0$ limit only the first term survives, and as will discuss below one recovers the Coulomb branch result. Since $k \geq |m|$ the vevs of maximally operators only receive contributions from the first term. Our explicit computations go up to dimension four, but if one assumes this structure persists in the vevs of higher dimension operators then the result (3.58) holds for maximally charged operators of all dimension. We will find that the field theory result does indeed reproduce (3.58) for all k, thus verifying this hypothesis. By contrast the vevs of non-maximally charged operators do receive other contributions and thus one needs to calculate explicitly the appropriate coefficients $(\alpha_{km}, \beta_{km}, \ldots)$.

These expressions make manifest the limiting behavior as $a \to 0$ and the theory passes to that of the Coulomb branch of $\mathcal{N} = 4$ on $R^{3,1}$. The R-charge and the energy as given in (3.54) vanish in this limit, as expected for supersymmetric vacua of $\mathcal{N} = 4$ on $R^{3,1}$. However, the scalar chiral primary vevs remain non-trivial for appropriate density functions $\rho(r, \phi)$. Each density function describing a regular bubbling geometry consists of N droplets d_i, such that $\rho(r, \phi)$ takes the value $1/\pi a^2$ on the droplet, and the area of each droplet is πa^2. Suppose the boundary of the droplet is described by $r = r_i + d_i(\phi)$, with r_i constant and some suitable function $d_i(\phi)$. Then the density function describing the droplet is

$$\rho_{d_i}(r, \phi) = \frac{1}{\pi a^2}\theta(r_i + d_i(\phi) - r), \tag{3.63}$$

such that

$$\int_{d_i} \rho_{d_i}(r, \phi) r dr d\phi = \frac{1}{N}. \tag{3.64}$$

Coulomb branch solutions are then obtained in the limit that r_i stays finite as $a \to 0$: the density function for each droplet behaves as

$$\rho_{d_i}(r, \phi) \to \frac{1}{N}\delta(x^1 - x_i^1)\delta(x^2 - x_i^2), \tag{3.65}$$

satisfying (3.64). Here (x_i^1, x_i^2) describe the location of the droplet in the 1-2 plane, and in this limit each of the N droplets is associated with an eigenvalue of the matrices (X^1, X^2). Clearly in the $a \to 0$ limit the disc density function describing the conformal vacuum becomes a delta function localized at the origin, $\rho(r, \phi) \to \delta(x^1)\delta(x^2)$.

Now taking the $a \to 0$ limit in the vevs of the scalar chiral primaries one gets

$$\begin{aligned} \langle \mathcal{O}_{S^{km}} \rangle &= \frac{N^2}{\pi^2}\frac{(k-2)}{2^{k/2}(k+1)}\sqrt{\frac{(k-1)}{k}}(\Delta\Phi)_{km}; \\ &= \frac{N^2}{\pi^2}2^{k/2}(k-2)\sqrt{\frac{(k-1)}{k}}\int_{R^2} dx^1 dx^2 \rho(x^1, x^2)(C^{km}_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k}), \end{aligned} \tag{3.66}$$

in exact agreement with the Coulomb branch vevs given in [2], restricting to an SO(4) invariant distribution.

4. Dual description

In this section we will consider half BPS states in $\mathcal{N} = 4$ SYM, and their relation to free fermions. We discuss the correspondence between an arbitrary half BPS state and a two-dimensional density distribution, which in turn is to be identified with the defining density function of the bubbling supergravity solution. In particular, we show that the state is not completely determined by this density distribution, but the density distribution does determine uniquely the vevs of all single trace chiral primary operators. These in turn are precisely the information that is encoded in the asymptotics of the LLM solutions. Thus one would anticipate that the LLM solutions receive higher order corrections, involving information beyond the density function, which capture the dual state uniquely.

4.1 Half BPS states in $\mathcal{N} = 4$ SYM

There is a one-to-one correspondence between half BPS SO(4) symmetric representations of $\mathcal{N} = 4$ SYM and symmetric polynomials in the eigenvalues of a complex matrix Z or Schur polynomials. Here Z is one combination of the six Hermitian scalars X^m of $\mathcal{N} = 4$ SYM, given by $Z = X^1 + iX^2$.

There are several choices of basis for the gauge invariant multi-trace polynomials of Z:

(i) The **trace basis** of products of traces of Z is an obvious gauge invariant basis. For the group $\mathrm{U}(N)$ the multitraces can be labelled by $p(n)$ conjugacy classes of the permutation group S_n where $p(n)$ is the number of partitions of n. Labeling representatives of different conjugacy classes of S_n by σ_I, the basis of multi-trace operators is given by $\mathrm{Tr}(\sigma_I Z)$:

$$\mathrm{Tr}(\sigma_I Z) = \sum_{j_1 \cdots j_n} Z^{j_1}_{j_{\sigma_I(1)}} Z^{j_2}_{j_{\sigma_I(2)}} \cdots Z^{j_n}_{j_{\sigma_I(n)}}. \tag{4.1}$$

For $\mathrm{SU}(N)$ Z is traceless and one must therefore restrict to elements of S_n without 1-cycles; the distinction between $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ is however not important in the $N \to \infty$ limit relevant here.

(ii) The **Schur polynomial basis** is, in the case of $\mathrm{U}(N)$, a sum over these trace operators, weighted by the characters of σ in the representation R of S_n, namely

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}(\sigma Z). \tag{4.2}$$

The representations R can be labeled by Young diagrams with n boxes, which correspond to partitions of n and there are thus $p(n)$ Schur polynomials of degree n. An advantage of this basis is that the two-point functions are diagonal. Again modifications are needed for the case of $\mathrm{SU}(N)$, but these give $1/N$ effects which will not be relevant here.

Note that another useful basis is the dual basis, dual to the trace basis, but this will not play a role here.

An arbitrary half BPS state $|\Phi\rangle$ preserving SO(4) R symmetry can therefore be written as a superposition of states

$$|\Phi\rangle = \sum_R a_R \chi_R(Z)|\Omega\rangle = \sum_I b_I \mathrm{Tr}(\sigma_I Z)|\Omega\rangle, \tag{4.3}$$

for suitable (complex) coefficients a_R and b_I, with $|\Omega\rangle$ being the conformal vacuum. Denoting by $\mathcal{O}^{\mathcal{A}}$ the set of gauge invariant operators, the vevs of these operators in the state $|\Phi\rangle$ are given by

$$\begin{aligned}\langle \mathcal{O}^{\mathcal{A}} \rangle_\Phi &= \sum_{R,R'} a_R^* a_{R'} \langle \Omega | (\chi_R(Z))^\dagger \mathcal{O}^{\mathcal{A}} \chi_{R'}(Z) | \Omega \rangle; \\ &= \sum_{I,J} b_I^* b_J \langle \Omega | (\mathrm{Tr}(\sigma_I Z))^\dagger \mathcal{O}^{\mathcal{A}} \mathrm{Tr}(\sigma_J Z) | \Omega \rangle.\end{aligned} \tag{4.4}$$

A state is an eigenstate of the dilatation operator and of the R-symmetry (in the 1-2 directions) with eigenvalue n if and only if the superposition involves only S_n. That is, only operators involving n fields Z are included in the superposition.

It is also important to note that in the $N \to \infty$ limit there are considerable simplifications in three point functions appearing in (4.4). Let us consider first computations in the trace basis, where the operators are normalized as $C_{\sigma_n^I} \mathrm{Tr}(\sigma_I^n Z)$ with n the dimension. The normalization factors $C_{\sigma_n^I}$ are such that the basis is orthonormal in the large N limit, namely

$$C_{\sigma_n^I} C_{\sigma_m^J} \langle \Omega | (\mathrm{Tr}(\sigma_I^n Z))^\dagger \mathrm{Tr}(\sigma_J^m Z) || \Omega \rangle = \delta_{IJ} \delta^{nm} + \mathcal{O}(1/N), \tag{4.5}$$

and the large N scaling of $C^2_{\sigma_n^I}$ is $1/N^n$.

Now consider the three point functions (4.4) in which the operators $\mathcal{O}^{\mathcal{A}}$ are single trace operators built from the six scalar fields X^m. These are clearly the relevant operators to compare with the holographic results. As discussed in [36] three point functions which are extremal, so that the conjugate operator has a dimension which is the sum of the dimensions of the other operators, and those which are non-extremal are known to have different large N behavior. Since the state $|\Phi\rangle$ is a sum of terms each of which is maximally charged, i.e. it has $j = \Delta$, it follows that the 3-point functions are never extremal when $\mathcal{O}^{\mathcal{A}}$ is not maximally charged. Moreover the large N behavior depends on whether the other operators in the correlator are single or multi-trace. We discuss in appendix C the large N behavior of such correlators, and summarize here the relevant results:

Non-extremal correlators: non-extremal three point functions for which $\mathcal{O}^{\mathcal{A}}$ are (orthonormal) single trace operators scale as $1/N$ or smaller in the large N limit. Note that this assumes that the dimensions of all operators in the correlator are small compared to N. In the case that the operator $\mathcal{O}^{\mathcal{A}}$ is neutral under $\mathrm{SO}(2) \times \mathrm{SO}(4)$ R symmetry the correlators behave as $1/N$ only for the diagonal terms, namely when $\sigma_I^n = \sigma_J^m$. When the operator $\mathcal{O}^{\mathcal{A}}$ has a non-maximal SO(2) charge m the correlators involving single trace operators still behave as $1/N$; correlators involving multi-trace operators are generically subleading in N, but for special cases can also behave as $1/N$.

Extremal correlators: extremal three point functions scale as one or smaller in the large N limit. Correlators involving only single trace operators behave as $1/N$, whilst multi-trace correlators in which $\mathrm{Tr}(\sigma_I^n Z) = \mathcal{O}^{\mathcal{A}} \mathrm{Tr}(\sigma_J^m Z)$ (and therefore σ_I^n is necessarily multi-trace) are of order one.

One can also rephrase these results in terms of the Schur polynomial basis, as discussed in [6]. As we will show in section 6.1, the vev of the SO(2) $\times$ SO(4) neutral operators in the state built from a given Schur polynomial of dimension n is independent of the choice of Schur polynomial. To leading order in N it behaves as n/N. This result has an immediate corollary: consider geometries dual to different superpositions of the Schur polynomials, all of the same dimension n. Then symmetry implies only the neutral operators acquire vevs but these vevs differ only by $1/N$ effects, so these geometries are not reliably distinguishable within supergravity.

Now consider a superposition of states of different dimension (and thus R charge). In such a case SO(2) charged single trace operators acquire vevs, and the computation of the vevs of maximally charged operators necessarily involves extremal correlators. The N scalings of these vevs depend crucially on the specific Schur polynomials, or equivalently multi-trace operators, appearing in the superposition. Superpositions involving single trace operators will lead to vevs which are suppressed by $1/N$ relative to multi-trace superpositions, and thus these are immediately distinguishable. An explicit example illustrating this effect will be discussed in section 6.3.

4.2 Relation to free fermions

Consider irreducible representations of the symmetry group S_n. These may be characterized by a sequence of non-negative integers $\{\lambda\} = (\lambda_1, \ldots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ and $\sum_{i=1}^{N} \lambda_i = n$. The sequence defines a Young tableau with the number of boxes in the ith row being λ_i and the total number of boxes being n; let $\chi^n_{\{\lambda\}}(Z)$ be the corresponding Schur polynomial. In this section we will review the relationship between Schur polynomials and free fermions.

We introduce a second quantized free fermion field

$$\Psi(z, z^*, t) = \sum_{l=0}^{\infty} \hat{C}_l e^{-i(l+1)t} \Phi_l(z, z^*) \tag{4.6}$$

where $(\hat{C}_l, \hat{C}_l^\dagger)$ satisfy the anti-commutation relation $\{\hat{C}_l, \hat{C}_m^\dagger\} = \delta_{lm}$. Note that throughout this section we will set the inverse radius a of the S^3 to one; this sets the mass scale in the matrix model, and thus of the fermions, to one. The functions $\Phi_l(z, z^*)$ are the orthonormal wavefunctions of the lowest Landau level, and are given by

$$\Phi_l(z, z^*) = \sqrt{\frac{2^{l+1}}{\pi l!}} z^l e^{-zz^*}. \tag{4.7}$$

The fermion field $\Psi(z, z^*, t)$ satisfies the constraint that the total number of fermions be N,

$$\int dz dz^* \Psi^\dagger(z, z^*, t) \Psi(z, z^*, t) = \sum_{l=0}^{\infty} \hat{C}_l^\dagger \hat{C}_l = N. \tag{4.8}$$

The ground state is denoted $|\Omega\rangle$ and is given by

$$|\Omega\rangle = \hat{C}^\dagger_{N-1} \hat{C}^\dagger_{N-2} \cdots \hat{C}^\dagger_1 \hat{C}^\dagger_0 |0\rangle, \tag{4.9}$$

where $|0\rangle$ is the Fock vacuum defined by $\hat{C}_l|0\rangle = 0$ for all l. We will denote by $|\Phi\rangle$ a generic state containing N fermions; each such state can also be expressed as a superposition of Schur polynomials. The Schur polynomial $\chi^n_{\{\lambda\}}(Z)$ corresponds to the state

$$\hat{C}^\dagger_{N-1+\lambda_1}\hat{C}^\dagger_{N-2+\lambda_2}\cdots\hat{C}^\dagger_{1+\lambda_{N-1}}\hat{C}^\dagger_{\lambda_N}|0\rangle. \tag{4.10}$$

Now consider the expectation value of the density function defined as

$$\hat{U}(z,z^*,t) = \Psi^\dagger(z,z^*,t)\Psi(z,z^*,t). \tag{4.11}$$

In the conformal vacuum

$$\begin{aligned}\langle\hat{U}(z,z^*,t)\rangle_\Omega &= \sum_{l=0}^{N-1}\Phi^*_l(z,z^*)\Phi_l(z,z^*); \qquad (4.12)\\ &= \sum_{l=0}^{N-1}\frac{2^{l+1}}{\pi l!}(zz^*)^l e^{-2zz^*} \equiv 2\pi^{-1}e^{-2zz^*}E_{N-2}(2zz^*),\end{aligned}$$

where by definition

$$E_{N-1}(y) = \sum_{l=0}^{N}\frac{y^l}{l!}. \tag{4.13}$$

For $N \gg 1$,

$$e^{-y}E_{N-1}(y) \to \theta(N-y) + f(y), \tag{4.14}$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. The function $f(y)$ describes the smearing of the step function; $f(y)$ has support only within a region around $y = N$ of width of order one and is such that $f(y) < 0$ for $y < N$, $f(y) > 0$ for $y > N$ with

$$\int_0^\infty dy f(y) = 0; \qquad \int_0^\infty dy|f(y)| = \mathcal{O}(1). \tag{4.15}$$

Thus to leading order as $N \to \infty$

$$\langle\hat{U}(z,z^*)\rangle_\Omega = \frac{2}{\pi}\theta(N-2|z|^2). \tag{4.16}$$

To compare with the supergravity results one therefore needs

$$|z| = \sqrt{\frac{N}{2}}|w|; \qquad \langle\hat{U}\rangle_\Omega = 2\rho, \tag{4.17}$$

with $|w|$ identified with the supergravity coordinate r. In a generic state $|\Phi\rangle$ the density function is given by

$$\begin{aligned}\langle\hat{U}(z,z^*,t)\rangle_\Phi &= \sum_{l,m}\Phi^*_l(z,z^*)\Phi_m(z,z^*)e^{i(l-m)t}\langle\hat{C}^\dagger_l\hat{C}_m\rangle_\Phi; \qquad (4.18)\\ &= \sum_{l,m}e^{i(l-m)t}(z^*)^l z^m\sqrt{\frac{2^{2+l+m}}{\pi^2 l!m!}}e^{-2z^*z}U^\Phi_{lm},\end{aligned}$$

where we define

$$U^\Phi_{lm} = \langle\hat{C}^\dagger_l\hat{C}_m\rangle_\Phi. \tag{4.19}$$

The supergravity density function is related to this via

$$\langle\hat{U}(t=0)\rangle_\Phi = 2\rho. \tag{4.20}$$

4.3 Extracting the state from a distribution

In this section we consider how to derive the specific superposition of Schur polynomials corresponding to a given distribution. Recall that the expectation value of the density function given in (4.18) is

$$\langle \hat{U}(z, z^*, t)\rangle_\Phi = \sum_{l,m} \Phi_l^*(z, z^*)\Phi_m(z, z^*)e^{i(l-m)t}\langle \hat{C}_l^\dagger \hat{C}_m\rangle_\Phi. \tag{4.21}$$

Thus by integrating a given density function with respect to a suitable basis of orthonormal polynomials one can extract the coefficients $\langle \hat{C}_l^\dagger \hat{C}_m\rangle_\Phi$. Note however that these expansion functions $\Phi_l^*\Phi_m$ are not orthogonal, when integrated over the plane with unit measure. So in practice it is actually more convenient to work with the density function in phase space, $\hat{u}(p, q, t)$, which is naturally expanded in a useful basis of orthonormal functions. The explicit relationship between the density functions $\hat{U}(z, z^*, t)$ and $\hat{u}(p, q, t)$ was given in [10]:

$$\hat{U}(z, z^*, t) = \int \frac{d\Lambda d\Lambda^*}{4\pi^2} e^{-\Lambda^* z+\Lambda z^* -\frac{1}{4}\Lambda\Lambda^*} \int dpdq e^{-\Lambda(q+ip)+\Lambda^*(q-ip)}\hat{u}(p, q, t). \tag{4.22}$$

Then, following [10] one finds that

$$\int dzdz^*(-1)^k(z^*)^j z^k \hat{U} = \frac{\partial^{j+k}}{\partial\Lambda^j\partial\Lambda^{*k}} \left(e^{-\frac{1}{4}\Lambda\Lambda^*} \int dpdq e^{-\Lambda(q+ip)+\Lambda^*(q-ip)}\hat{u} \right)_{\Lambda=\Lambda^*=0} \tag{4.23}$$

Now to make manifest the behavior in the large N limit one should rescale these coordinates as in (4.17) so that

$$|z| = \sqrt{\frac{N}{2}}|w|; \qquad q = \sqrt{\frac{N}{2}}x; \qquad p = \sqrt{\frac{N}{2}}y. \tag{4.24}$$

Retaining only the leading order terms in (4.23) as $N \to \infty$ gives

$$\int d^2w(w^*)^j w^k \hat{U} = \int d^2x r^{j+k} e^{i(k-j)\phi}\hat{u}, \tag{4.25}$$

where $x = r\cos\phi$, $y = r\sin\phi$. As we have seen from the holographic computations, and will discuss below, all one point functions are expressed in terms of these integrals. Thus at leading order in N one can identify $|w| = r$ and the difference between the distributions $(\hat{U}, \hat{u})$ is not visible. Whilst it is more natural for the droplet distribution in the bulk solution to be identified with the phase space distribution, rather than the z space distribution, this is not distinguishable at leading order in N.

In some calculations, such as that of one point functions which we will discuss below, it is more convenient to use the z space distribution, and exploit the simple form of its expansion in exponentials. For the current purpose it is rather more convenient to work with the phase space distribution, since this is expanded in a natural basis of orthonormal functions, the Laguerre polynomials. That is, the phase space distribution is given by [10]:

$$\begin{aligned} \pi\rho_\Phi = & \sum_{m\le n} \sqrt{\frac{m!}{n!}}(-1)^m \chi^{\frac{1}{2}(n-m)} e^{-\frac{1}{2}\chi} e^{i(m-n)\phi} L_m^{n-m}(\chi)\langle \hat{C}_m^\dagger \hat{C}_n\rangle_\Phi \\ & + \sum_{m>n} \sqrt{\frac{n!}{m!}}(-1)^n \chi^{\frac{1}{2}(m-n)} e^{-\frac{1}{2}\chi} e^{i(m-n)\phi} L_n^{m-n}(\chi)\langle \hat{C}_m^\dagger \hat{C}_n\rangle_\Phi, \end{aligned} \tag{4.26}$$

where $\chi = Nr^2$ and $L_m^{n-m}(\chi)$ is the Laguerre polynomial defined by

$$L_n^\alpha(\chi) = \sum_{p=0}^{n} (-1)^p \begin{pmatrix} n+\alpha \\ n-p \end{pmatrix} \frac{\chi^p}{p!}, \tag{4.27}$$

for which the orthogonality relation is

$$\int_0^\infty d\chi e^{-\chi} \chi^\alpha L_n^\alpha(\chi) L_m^\alpha(\chi) = \delta_{mn} \frac{(n+\alpha)!}{n!}, \tag{4.28}$$

for integral α. In the conformal vacuum one gets

$$\pi\rho_\Omega = \sum_{m=0}^{N-1} (-)^m e^{-\frac{1}{2}\chi} L_m(\chi) \tag{4.29}$$

where $L_m(\chi) \equiv L_m^0(\chi)$. Using the identity

$$\int_0^\infty d\chi e^{-\frac{1}{2}\chi} L_m(\chi) = 2(-1)^m \tag{4.30}$$

one can show that this satisfies the normalization condition $\int d^2x\rho_\Omega = 1$. Moreover for large N the distribution asymptotes as before to a disc, $\pi\rho_\Omega \to \theta(1-r)$.

Using the orthogonality relation for the Laguerre polynomials one can now extract the $\langle \hat{C}_m^\dagger \hat{C}_n \rangle_\Phi$ via:

$$\begin{aligned} \langle \hat{C}_m^\dagger \hat{C}_{m+p} \rangle_\Phi &= (-1)^m \frac{1}{4} \sqrt{\frac{(m+p)!}{m!}} \int d\phi e^{ip\phi} d\chi \chi^{p/2} e^{-\chi/2} L_m^p(\chi) \rho_\Phi; \\ \langle \hat{C}_{m+p}^\dagger \hat{C}_m \rangle_\Phi &= (-1)^m \frac{1}{4} \sqrt{\frac{(m+p)!}{m!}} \int d\phi e^{-ip\phi} d\chi \chi^{p/2} e^{-\chi/2} L_m^p(\chi) \rho_\Phi, \end{aligned} \tag{4.31}$$

where $(m,p) \geq 0$.

Thus from the distribution ρ_Φ one can extract the complete set of $\langle \hat{C}_m^\dagger \hat{C}_n \rangle_\Phi$. Let us now discuss whether knowledge of these is in principle sufficient to determine the state $|\Phi\rangle$.[5] Consider first the case where $|\Phi\rangle$ has definite dimension n, so that

$$|\Phi\rangle = \sum_{\{\lambda\}} a_{\{\lambda\}} |n; \{\lambda\}\rangle. \tag{4.32}$$

Normalization of the state implies that $\sum_{\{\lambda\}} |a_{\{\lambda\}}|^2 = 1$. In such a state $\langle \hat{C}_m^\dagger \hat{C}_p \rangle_\Phi$ is non-zero only for $m = p$ and

$$\langle \hat{C}_m^\dagger \hat{C}_m \rangle_\Phi = \sum_{\{\lambda\}} |a_{\{\lambda\}}|^2 \delta_{\{\lambda\}m}, \tag{4.33}$$

where $\delta_{\{\lambda\}m} = 1$ iff the corresponding state contains a fermion at level m. Therefore one cannot extract the phases of $a_{\{\lambda\}}$ from this information: the density function is not sufficient to completely determine the state. There is one exception to this: when precisely

[5]Related discussions appeared in the recent paper [38].

N of the $\langle \hat{C}_m^\dagger \hat{C}_m \rangle_\Phi$ are non-zero and equal to one, the corresponding state is necessarily a single Schur polynomial. In this case the summation in (4.32) collapses to one term, and the overall phase of the state plays no role. Note that this is precisely the case that was discussed in [37], but for a general state of definite dimension the distribution is not sufficient to determine the state. To determine the phases in the general case one would need to know in addition

$$\langle \prod_i \hat{C}_{m_i}^\dagger \prod_j \hat{C}_{m_j} \rangle_\Phi, \qquad \sum_i m_i = \sum_j m_j. \tag{4.34}$$

It will be made manifest in the next section that $\langle \hat{C}_m^\dagger \hat{C}_p \rangle_\Phi$ determines the expectation values of single trace operators in the state, see (5.32), whilst (4.34) is related to the expectation values of multi-trace (neutral) operators.

Note that the discussion so far has made no restriction on N. Even if one can determine the $\langle \hat{C}_m^\dagger \hat{C}_m \rangle_\Phi$ exactly one can still not determine the state. Of course when one takes the $N \to \infty$ limit and sharpens the distribution such that it gives a regular supergravity solution the situation will be worse. One will not be able to determine the coefficients $\langle \hat{C}_m^\dagger \hat{C}_m \rangle_\Phi$ exactly, and thence one can only determine the leading behavior in N of the $|a_{\{\lambda\}}|^2$.

Now consider a general state $|\Phi\rangle$ which does not have a definite dimension, so that

$$|\Phi\rangle = \sum_{n,\{\lambda\}} a_{n,\{\lambda\}} |n; \{\lambda\}\rangle. \tag{4.35}$$

The neutral vevs $\langle \hat{C}_m^\dagger \hat{C}_m \rangle_\Phi$ still do not determine the phases of the $a_{n,\{\lambda\}}$. However, some phase information is obtained via the vevs of charged operators. That is,

$$\begin{aligned} \langle \hat{C}_{m+p}^\dagger \hat{C}_m \rangle_\Phi &= \sum_{n,\{\lambda\}} a^*_{n+p,\{\lambda\}_p} a_{n,\{\lambda\}} \langle n+p; \{\lambda\}_p | \hat{C}_{m+p}^\dagger \hat{C}_m | n; \{\lambda\}\rangle, \\ &= \sum_{n,\{\lambda\}} a^*_{n+p,\{\lambda\}_p} a_{n,\{\lambda\}}, \end{aligned} \tag{4.36}$$

where the Schur polynomial $|n+p; \{\lambda\}_p\rangle$ is precisely $\hat{C}_{m+p}^\dagger \hat{C}_m |n; \{\lambda\}\rangle$. That is, the state $|n+p; \{\lambda\}_p\rangle$ differs from $|n; \{\lambda\}\rangle$ by only one fermion. So only a subset of the phase information is obtained; this is sufficient to determine the state when the superposition contains Schur polynomials such that each of which differs by only one fermion from at least one other polynomial in the superposition. If however the state contains at least one Schur polynomial which differs by two fermions or more from all other Schur polynomials in the superposition, then the phase of the coefficients of these terms cannot be determined without vevs of multi-trace operators. Thus for a general distribution one would again need vevs of multi-trace operators to determine the coefficients in $|\Phi\rangle$ uniquely.

The supergravity solutions are constructed entirely out of the density function ρ_Φ and therefore contains information only about the expectation values of single trace operators. To determine the state $|\Phi\rangle$ one needs the expectation values of all other operators, which are not determined by ρ_Φ. Therefore, one would expect that the higher order corrections to the LLM bubbling solution, apart from correcting the distribution ρ_Φ, would also involve additional information so that the corrected solution captures the entire vacuum structure. This is in line with the fact that the IIB supersymmetry rules are expected to receive non-trivial higher derivative corrections.

5. Computation of vevs

In the previous section we set up the correspondence between a general half BPS state and a density distribution. The information extracted holographically is the vevs of chiral primary operators, and in this section we will discuss how these vevs may be computed in an arbitrary half BPS state. We compute explicitly vevs of all operators up to dimension four, and the vevs of maximally charged operators of arbitrary dimension, and find exact agreement with the holographic results. These results are a detailed confirmation of the correspondence between LLM bubbling solutions and 1/2 BPS states, and moreover provide evidence that the vevs are not renormalized, as one might anticipate given the sixteen preserved supercharges.

5.1 Energy and R-charge

In a generic state $|\Phi\rangle$ the energy and R-charge J relative to that of the conformal vacuum is

$$(E - E_c) = J = \sum_m m\langle \hat{C}^\dagger_m \hat{C}_m\rangle_\Phi - \sum_{m=0}^{N-1} m = \sum_m m\langle \hat{C}^\dagger_m \hat{C}_m\rangle_\Phi - \frac{1}{2}N(N-1), \tag{5.1}$$

where $E_c = 3N^2/16$ is the Casimir energy on $R\times S^3$. This Casimir is clearly not reproduced correctly by the matrix model, since there are contributions from all KK modes on the S^3 of all SYM fields. Now note that

$$\int d^2z|z|^{2k}\langle \hat{U}(z,z^*,t)\rangle_\Phi = \sum_m \frac{(m+k)!}{2^k m!}\langle \hat{C}^\dagger_m \hat{C}_m\rangle_\Phi. \tag{5.2}$$

Thus

$$\sum_m m\langle \hat{C}^\dagger_m \hat{C}_m\rangle_\Phi = \int d^2z(2|z|^2-1)\langle \hat{U}(z,z^*,t)\rangle_\Phi, \tag{5.3}$$

and hence

$$\begin{aligned}(E - E_c) = J &= \int d^2z\left(2|z|^2 - \frac{1}{2}(N+1)\right)\langle \hat{U}(z,z^*,t)\rangle_\Phi; \\ &= N^2\int d^2w\left(|w|^2 - \frac{1}{2}\left(1+\frac{1}{N}\right)\right)\rho,\end{aligned} \tag{5.4}$$

which agrees with the holographic result (3.54), after taking the $N \to \infty$ limit.

5.2 Vevs of maximally charged operators

We now consider how the vevs of single trace operators $\mathcal{O}^{\Delta,k}$ of dimension Δ and SO(2) charge k may be computed. Here we will use $\mathcal{O}^{\Delta,k}$ to denote the operator in field theory, whose vevs we compute within free field theory, whilst $\mathcal{O}_{S^{\Delta k}}$ refers to the corresponding operator whose vevs at strong coupling were computed holographically.

Let us begin by computing the vev of the maximally charged single trace scalar operator of dimension k, $\mathcal{O}^{k,k}$, in a generic state. The operators $\mathcal{O}^{k,k}$ are implemented as follows:

$$\mathcal{O}^{k,k} = \mathcal{N}_k \lambda_{k,k} e^{ikt} \sum_{l=0}^{\infty} \sqrt{\frac{(l+k)!}{l!}} \hat{C}^{\dagger}_{l+k} \hat{C}_l. \tag{5.5}$$

The factor $\lambda_{k,k}$ is chosen such that the two point function satisfies the following normalization condition:

$$\langle (\mathcal{O}^{k_1,k_1})^{\dagger}(t_1) \mathcal{O}^{k_2,k_2}(t_2) \rangle = \mathcal{N}_{k_1}^2 \delta_{k_1 k_2}. \tag{5.6}$$

where $\mathcal{N}_k$ is defined in (B.4). $\mathcal{N}_k$ is the appropriate normalization for the two point functions extracted holographically. The normalization condition implies that

$$\lambda_{k,k}^{-2} = \sum_{N-k}^{N-1} \frac{(l+k)!}{l!} = \frac{1}{(1+k)} \left(\frac{(N+k)!}{(N-1)!} - \frac{N!}{(N-k-1)!} \right) \overset{N \to \infty}{\to} N^k k + \mathcal{O}(N^{k-1}). \tag{5.7}$$

The corresponding integral representation of the operators is therefore ($k \neq 1$)

$$\begin{aligned} \mathcal{O}^{k,k}(t) &= \mathcal{N}_k \lambda_{k,k} 2^{\frac{k}{2}} \int dz dz^* z^k \hat{U}(z, z^*, t); \\ &= \frac{N}{\pi^2 \sqrt{k}} \sqrt{k-1}(k-2) 2^{\frac{k}{2}} N^{-\frac{k}{2}} \int dz dz^* z^k \hat{U}(z, z^*, t). \end{aligned} \tag{5.8}$$

The expectation value of this operator in a generic state $|\Phi\rangle$ is then given by

$$\langle \mathcal{O}^{k,k}(t) \rangle_{\Phi} = \frac{N}{\pi^2 \sqrt{k}} \sqrt{k-1}(k-2) 2^{\frac{k}{2}} N^{-\frac{k}{2}} \int dz dz^* z^k \langle \hat{U}(z, z^*, t) \rangle_{\Phi}. \tag{5.9}$$

The integral may be rewritten using using (4.17), (4.20) as

$$\langle \mathcal{O}^{k,k}(t) \rangle_{\Phi} = \frac{N^2}{\pi^2 \sqrt{k}} \sqrt{k-1}(k-2) e^{ikt} \int d^2 w e^{ik\phi} |w|^k \rho, \tag{5.10}$$

in exact agreement with the holographic result (3.58).

Consider the $k = 1$ operator. Here the distinction between U(N) and SU(N) becomes important: the vanishing of the trace in the latter means that there is no dimension one operator, $\mathcal{O}^{1,1}$. This constraint can be incorporated here by restricting to configurations in which

$$\langle \mathcal{O}^{1,1} \rangle = 0 = \int d^2 w w \rho, \tag{5.11}$$

which is indeed the condition imposed on the holographic distribution.

5.3 Other scalar chiral primaries

Now let us consider the remaining scalar chiral primary operators $\mathcal{O}^{\Delta k}$, which are not maximally charged, $\Delta > |k|$. The action of such an operator on the conformal vacuum $|\Omega\rangle$ or any other 1/2 BPS state $|\Phi\rangle$ built from Schur polynomials creates a state which cannot be described in terms of Schur polynomials. The reason is that the other scalar operators contains not only the scalar fields $(Z, \bar{Z})$ but also the remaining four $\mathcal{N} = 4$ SYM scalar fields. The latter are not contained in the Schur polynomials, and thence not in the free fermion description.

Suppose however one wishes to compute one point functions of these scalar operators in a state $|\Phi\rangle$. To do so, following (4.4), one needs to know three point functions between such an operator and two maximally charged operators. Consider the computation of such a three point function in free field theory; at tree level the computation actually only involves the fields $(Z, \bar{Z})$. Take for example a three point function such as

$$\langle \mathrm{Tr}(\bar{Z})^k(x)\mathcal{O}^{2p,0}(y)\mathrm{Tr}(Z)^k(z)\rangle, \tag{5.12}$$

for which the single trace $\mathrm{SO}(2) \times \mathrm{SO}(4)$ singlet operator $\mathcal{O}^{2p,0}$ has the structure

$$\mathcal{O}^{2p,0} = a_1 \mathrm{Tr}\left((Z\bar{Z})^p + \cdots\right) + a_2 \mathrm{Tr}\left((Z\bar{Z})^{p-2}R^2 + \cdots\right) + \cdots, \tag{5.13}$$

where the ellipses within the trace denote cyclic permutations, $(a_1, a_2, \ldots)$ are constants and $R^2 = \sum_{i=1}^{4}(X_i)^2$ denotes collectively the other scalars X_i of $\mathcal{N} = 4$ SYM. Since the latter have no propagators with the fields $(Z, \bar{Z})$ and cannot be self-contracted, only the first term contributes in the three point function (5.13).

Therefore, one would anticipate being able to implement the scalar operators with free fermions such that one can compute such one point functions. One would not however expect to be able to compute two point functions, or general higher point functions of such operators, using the free fermion description.

5.3.1 Neutral operators

Suppose one implements the dimension two neutral operator as

$$\mathcal{O}^{2,0} = \mathcal{N}_2\lambda_{2,0}\sum_m \hat{C}_m^\dagger \hat{C}_m(m - \beta). \tag{5.14}$$

Then the normalization factor $\lambda_{2,0}$ and the constant β should be fixed such that the one point function of this operator vanishes in the conformal vacuum; the three point function of the operator with charged operators gives the correct $\mathcal{N} = 4$ results and the vev reduces to the Coulomb branch result as the radius of the S^3 is increased. Imposing the first constraint, $\langle \mathcal{O}^{2,0}\rangle_\Omega = 0$, implies that

$$\beta = N^{-1}\sum_{m=0}^{N-1} m = \frac{1}{2}(N - 1). \tag{5.15}$$

Note that this value for β implies that the operator $\mathcal{O}^{2,0}$ annihilates the conformal vacuum, $\mathcal{O}^{2,0}|\Omega\rangle = 0$, and therefore the two point function for this operator also vanishes. This makes manifest the point made above, that one can only obtain the vevs of neutral operators from the matrix model. The corresponding integral representation of the operator is

$$\mathcal{O}^{2,0} = 2\mathcal{N}_2\lambda_{2,0} \int dzdz^* \left(|z|^2 - \frac{1}{4}N \right) \hat{U}(z, z^*, t) \tag{5.16}$$

where subleading terms as $N \to \infty$ are dropped. The normalization $\lambda_{2,0}$ is fixed by taking the flat space limit: only the leading order term is retained, and comparison with the result (3.66) gives

$$\lambda_{2,0} = \frac{\sqrt{2}}{N\sqrt{3}}. \tag{5.17}$$

We have also checked explicitly that this result is consistent with $\mathcal{N} = 4$ three point functions, involving charged operators. Using (4.17), (4.20) one can rewrite the vev as

$$\langle \mathcal{O}^{2,0} \rangle = \frac{\sqrt{2}N^2}{\sqrt{3}\pi^2} \int d^2w \left(|w|^2 - \frac{1}{2} \right) \rho, \tag{5.18}$$

in exact agreement with the holographic result (3.59).

Now let us apply the same techniques to obtain expressions for the vev of the dimension four neutral operator. For the operator $\mathcal{O}^{4,0}$ one gets

$$\mathcal{O}^{4,0} = \frac{2\mathcal{N}_4}{\sqrt{5}N^2} \sum \hat{C}^\dagger_m \hat{C}_m \left(\frac{3m^2}{4} + b_1 m + b_2 \right). \tag{5.19}$$

Here the overall normalization is again fixed, so that the operator in the integral representation gives the correct expression in the Coulomb branch limit. Imposing the vanishing of the vev in the conformal vacuum implies that

$$2N^3 + N^2(4b_1 - 3) + N(8b_2 - 4b_1 + 1) = 0. \tag{5.20}$$

Calculating the three point function with single trace charged operators of dimension two such that $\hat{s}_2 = \mathcal{N}_2^{-1}\mathcal{O}^{2,2}$ gives

$$\langle \hat{s}^\dagger_2(t)\mathcal{O}^{4,0}\hat{s}_2(t') \rangle = \frac{2\mathcal{N}_4}{\sqrt{5}N} \lambda^2_{2,2}(6N^2 + 4Nb_1 + \cdots) = \frac{2\mathcal{N}_4}{N\sqrt{5}}, \tag{5.21}$$

where the ellipses denote terms which are subleading as $N \to \infty$. Then solving these equations to leading order in N gives

$$b_1 = -N; \quad b_2 = \frac{1}{4}N^2. \tag{5.22}$$

These values can be shown to also be consistent with other three point functions involving different charged operators. In integral representation this implies that

$$\mathcal{O}^{4,0} = \frac{2\mathcal{N}_4}{\sqrt{5}N^2} \int dzdz^* \left(3|z|^4 - 2N|z|^2 + \frac{1}{4}N^2 \right) \hat{U}(z, z^*, t), \tag{5.23}$$

where again only leading terms as $N \to \infty$ are retained, and using (4.17) this gives

$$\langle \mathcal{O}^{4,0} \rangle = \frac{\sqrt{3}N^2}{\sqrt{5}\pi^2} \int d^2w (3|w|^4 - 4|w|^2 + 1)\rho, \tag{5.24}$$

which agrees with the holographic result (3.59).

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5.3.2 Charged operators

Now let us treat the charged operators in a similar fashion. For the dimension three operator

$$\mathcal{O}^{3,1} = \frac{2\mathcal{N}_3}{N^{3/2}} \int dz dz^* z(|z|^2 + c_3)\hat{U}(z, z^*, t), \tag{5.25}$$

which implies in the fermion representation

$$\mathcal{O}^{3,1} = \mathcal{N}_3 \frac{1}{\sqrt{2}N^{3/2}} e^{it} \sum_m \left((m + 2 + 2c_3)(m+1)^{1/2} \right) \hat{C}^\dagger_{m+1} \hat{C}_m. \tag{5.26}$$

Now we use the three point function (B.8) to fix the coefficient c_3; to leading order in N this gives

$$c_3 = -\frac{1}{2}N, \tag{5.27}$$

and thus

$$\langle \mathcal{O}^{3,1} \rangle = \frac{N^2}{\pi^2} e^{it} \int d^2w w(|w|^2 - 1)\rho. \tag{5.28}$$

However the constraint (5.11) implies that the second term in (5.28) vanishes, and thus that the holographic result (3.59) is reproduced.

For the dimension four operator

$$\mathcal{O}^{4,2} = \frac{8\mathcal{N}_4}{\sqrt{10}N^2} \int dz dz^* z^2(|z|^2 + c_4)\hat{U}(z, z^*, t), \tag{5.29}$$

which implies in the fermion representation

$$\mathcal{O}^{4,2} = e^{2it} \frac{2\mathcal{N}_4}{\sqrt{10}N^2} \sum_m \left((m + 3 + 2c_4)\sqrt{(m+2)(m+1)} \right) \hat{C}^\dagger_{m+2} \hat{C}_m. \tag{5.30}$$

Again we use a three point function (B.8) to fix the coefficient c_4; to leading order in N this gives $c_4 = -\frac{1}{2}N$, and thus the vev is

$$\langle \mathcal{O}^{4,2} \rangle = \frac{4\sqrt{3}N^2}{\sqrt{10}\pi^2} e^{2it} \int d^2w w^2(|w|^2 - 1)\rho, \tag{5.31}$$

in agreement with (3.59).

Thus, to summarize, we have implemented all single trace operators of dimension Δ and SO(2) charge k quadratically in fermions as

$$\mathcal{O}^{\Delta,k} = N^{-\frac{1}{2}\Delta} \mathcal{N}_\Delta e^{ikt} \sum_{m=0}^{\infty} P^{\Delta,k}(m) \hat{C}^\dagger_{m+k} \hat{C}_m, \tag{5.32}$$

with $P^{\Delta,k}(m)$ fixed so as to give the correct normalization and three point functions of $\mathcal{N} = 4$ SYM. Up to dimension four, it was sufficient to use three point functions involving only single trace operators, but for higher dimension operators one might also have to use additional three point functions involving multi-trace operators. Rewriting the vevs of these operators as integrals over the distribution gives the general form for the holographic vevs (3.61) and explicit agreement for all operators up to dimension four and maximally charged operators of all dimension.

6. Correspondence between supergravity solutions and states

In this section we explore how much one can deduce about the dual state from a given regular supergravity solution, using the information about the vevs of chiral primaries. We have already argued that even the exact distribution function does not in general determine the state uniquely, and in this section we will see how a given sharpened distribution (which gives a regular supergravity solution) can correspond to a number of distinct exact distributions.

We will also note that non-singular supergravity solutions which break the SO(2) rotational symmetry are necessarily dual to infinite superpositions of Schur polynomials. Superpositions of a small number of Schur polynomials typically give rise to distributions which cannot be approximated by step functions and thus do not correspond to regular geometries. Thus the natural field theory bases for half BPS states, which use R charge eigenstates, are not the natural bases for describing regular bubbling geometries.

We illustrate this point by considering a disc distribution with a ripple deformation of frequency n. Using the chiral primary vevs we argue that such a distribution is given by a coherent superposition of single trace operators. This identification follows very naturally from earlier discussion of quantum Hall liquids: area preserving deformations of a disc droplet are naturally described by coherent superpositions of fermionic excitations, or equivalently in terms of a collective chiral boson description.

6.1 Radially symmetric distributions

In this section we consider half BPS states associated with superpositions of Schur polynomials of the same dimension. Such states are eigenstates of the dilatation and R charge, so only SO(2) neutral operators can acquire expectation values. The corresponding distributions therefore preserve the rotational symmetry in the plane.

A given Schur polynomial $\chi^n_{\{\lambda\}}(Z)$ corresponds according to (4.10) to a distribution

$$\begin{aligned}\langle \hat{U}(z,z^*,t)\rangle_{n,\lambda} &= \sum_{p=1}^{N} \Phi^*_{\lambda_p+N-p}(z,z^*,t)\Phi_{\lambda_p+N-p}(z,z^*,t); \\ &= \frac{2}{\pi}e^{-2|z|^2}\sum_{p=1}^{N}\frac{(2|z|^2)^{\lambda_p+N-p}}{(\lambda_p+N-p)!}.\end{aligned} \tag{6.1}$$

Given such a distribution is radially symmetric, only the stress energy tensor along with neutral operators acquire expectation values. The expectation value of the former is clearly independent of λ, and depends only on n. One can now show that the vevs of neutral operators with dimensions $2k \ll N$ also do not distinguish between different Schur polynomials, for $N \to \infty$. First note that

$$\begin{aligned}\int d^2 z|z|^{2k}\langle\Delta\hat{U}(z,z^*,t)\rangle_{n,\lambda} &= 2^{-k}\sum_{p=1}^{N}\left(\frac{(\lambda_p+N+k-p)!}{(\lambda_p+N-p)!}-\frac{(N+k-p)!}{(N-p)!}\right)\\ &= 2^{-k}\sum_{p=1}^{N}\frac{(N+k-p)!}{(N-p)!}(\psi(N+k-p)-\psi(N-p))\lambda_p+\cdots\\ &= 2^{-k}N^{k-1}kn+\cdots\end{aligned} \tag{6.2}$$

where $\Delta\hat{U}_{n,\lambda} = (\hat{U}_{n,\lambda}-\hat{U}_{\Omega})$ and $\psi(x)$ is the Digamma function and ellipses denote terms which are subleading as $N \to \infty$. Thus the leading term depends only on $\sum_{p=1}^{N}\lambda_p = n$, and not the specific Schur polynomial. The vevs of neutral operators can be expressed in terms of such integrals as

$$\langle\mathcal{O}^{2k,0}\rangle_{n,\lambda} = \frac{\mathcal{N}_{2k}}{N^k}\sum_{l=0}^{k-1} d_l \int d^2 z|z|^{2(k-l)} N^l \langle\Delta\hat{U}(z,z^*,t)\rangle_{n,\lambda}, \tag{6.3}$$

for certain coefficients d_l. (In the previous sections we gave the d_l explicitly for $k=1,2$.) Using (6.2), one finds that

$$\langle\mathcal{O}^{2k,0}\rangle_{n,\lambda} = \frac{\mathcal{N}_{2k}n}{N}C_k, \qquad C_k = \sum_{l=0}^{k-1} d_l 2^{-(k-l)}(k-l), \tag{6.4}$$

regardless of the choice of λ. Note that this behavior concurs with the explicit result for the vev in the state created by the single trace operator $\mathrm{Tr}(Z^n)$, given in (B.6), and the latter result determines that

$$C_k = \frac{\sqrt{2k}}{2^{k-1}\sqrt{2k+1}}. \tag{6.5}$$

Thus the vevs of neutral operators in an R-charge eigenstate are at leading order in N independent of the specific choice of Schur polynomial superposition creating that state.

Expressed in the coordinates appropriate for comparing with supergravity, the density distribution takes the form

$$\rho_{n,\lambda} = \frac{1}{\pi}e^{-N|w|^2}\sum_{p=1}^{N}\frac{(N|w|^2)^{\lambda_p+N-p}}{(\lambda_p+N-p)!}, \tag{6.6}$$

and has the properties

$$\int d^2 w \rho_{n,\lambda} = 1; \qquad \int d^2 w|w|^2(\rho_{n,\lambda}-\rho_{\Omega}) = \frac{n}{N^2}. \tag{6.7}$$

The latter implies that the excess energy relative to the conformal vacuum is n, independently of λ. This density function is such that $0 \le \pi\rho_{n,\lambda} \le 1$ everywhere; however, as for the density function describing the conformal vacuum, $(\pi\rho_{n,\lambda})$ does not take the values $\{0,1\}$ everywhere, and therefore the corresponding supergravity solution constructed from $\rho_{n,\lambda}$ would be singular. Just as for the conformal vacuum, though, the density function can

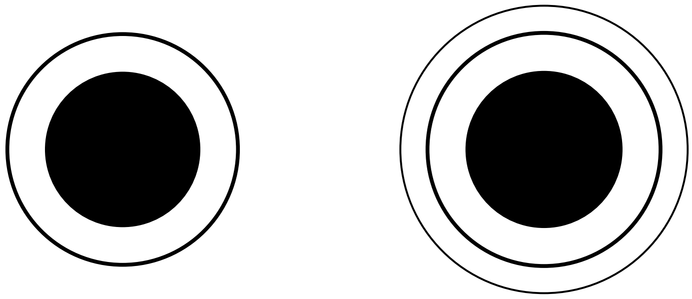

Figure 1: Sharpened distributions describing Schur polynomials of the same dimension $n \ll N$ cannot be distinguished by the corresponding vevs at leading order in N.

be written as a sum of theta functions plus correction terms describing the smearing which are subleading as $N \to \infty$. Retaining only the former leads to a non-singular supergravity solution.

For example, suppose one considers the Schur polynomial for which $\lambda_1 = n+1$ with $\lambda_p = 0$ otherwise; this corresponds to the state $\hat{C}^\dagger_{N+n}\hat{C}_{N-1}|\Omega\rangle$. The density function (6.6) in this case consists of a smoothed disc of radius one, along with a second peak localized around $|w|^2 = (1+\frac{n+1}{N})$. Now suppose one sharpens distribution so as to get a regular supergravity solution, consisting of a disc plus an annulus:

$$\begin{aligned} \rho = \frac{1}{\pi}: &\quad 0 \le |w|^2 \le \left(1 - \frac{1}{N}\right); &\quad \left(1+\frac{n}{N}\right) \le |w|^2 \le \left(1 + \frac{n+1}{N}\right), \\ \rho = 0: &\quad \left(1-\frac{1}{N}\right) < |w|^2 < \left(1+\frac{n}{N}\right); &\quad |w|^2 > \left(1+\frac{n+1}{N}\right). \end{aligned} \tag{6.8}$$

The smearing of the disc and the annulus to obtain the exact density function is described by correction terms which are subleading as $N \to \infty$.

As a second example, consider the Schur polynomial for which $\lambda_1 = n_1$ and $\lambda_2 = n_2 = n+1-n_1$, corresponding to $\hat{C}^\dagger_{N+n_1-1}\hat{C}^\dagger_{N+n_2-2}\hat{C}_{N-1}\hat{C}_{N-2}|\Omega\rangle$. Assuming that n_1 and n_2 differ by a finite amount, the density function (6.6) consists of a smooth disc, along with two localized peaks. (If n_1 and n_2 are comparable, these two peaks merge, to look like the one.) One can then obtain a corresponding supergravity solution by taking the density distribution to consist of a disk plus two annuli:

$$\begin{aligned} \rho = \frac{1}{\pi}: \qquad & 0 \le |w|^2 \le \left(1-\frac{2}{N}\right); \quad \left(1+\frac{(n_2-2)}{N}\right) \le |w|^2 \le \left(1+\frac{(n_2-1)}{N}\right), \\ & \left(1+\frac{(n_1-1)}{N}\right) \le |w|^2 \le \left(1+\frac{n_1}{N}\right). \end{aligned} \tag{6.9}$$

Both configurations we have described have the same energy $(n+1)$, and by the arguments above also have the same one point functions at leading order in N.

More generally, for an arbitrary Schur polynomial one can obtain a corresponding supergravity solution by sharpening the distribution into a set of annuli, as illustrated in figure 1. However, there is clearly not a unique map from such a set of annuli to

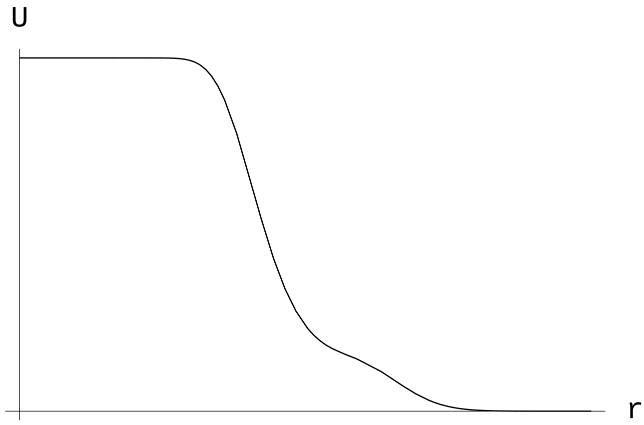

Figure 2: Distribution for a typical Schur polynomial; there is no distinct peak. The figure shows $N = 100$, $n = 30$, with a random distribution of λ_p.

a given Schur polynomial: Schur polynomials which are very similar to each other, and superpositions of similar Schur polynomials, give density distributions which can only be distinguished at subleading order in N. That is, the associated sharpened supergravity distributions, which consist only of annuli, can be the same. To give an example, consider a specific superposition of Schur polynomials of the same dimension

$$|\Phi\rangle = \sum_{k=0}^{n} a_k \hat{C}^\dagger_{N+n-k} \hat{C}^\dagger_{N-1-k} |\Omega\rangle, \tag{6.10}$$

with $\sum_{k=0}^{n} |a_k|^2 = 1$. When only one coefficient a_k is non-zero this collapses to the case discussed above of a single Schur polynomial, for which the sharpened distribution is a disc plus annulus. However, many other superpositions, such as those for which one a_k is much greater than the rest, will give precisely the same sharpened distribution. Moreover, a typical Schur polynomial for which $1 \ll n \ll N$ in which many of the λ_p are non-zero and different can give rise to a distribution which does not approximate a disk plus annuli: generically there are no strong peaks at $|w| > 1$, as illustrated in figure 2. By contrast a Schur polynomial for which the λ_p are equal does give rise to a strong peak, see figure 3. For more discussions on these issues, see [11, 37] along with the recent paper [38].

Note furthermore that a giant graviton, which is also a Schur polynomial of given dimension, necessarily corresponds to a radially symmetric distribution and not a disc with an excited droplet illustrated in figure 5, as suggested in some earlier papers. Indeed the maximal giant graviton, which has dimension N and is a Schur polynomial for which all $\lambda_p = 1$, corresponds to a distribution which approximates a disc with a hole in the middle, as illustrated in figure 4.

In the above discussion we have focused on Schur polynomials with dimension less than N. A Schur polynomial with dimension greater than or equal to N can never be represented by a single trace component, since there is of course no single trace operator with such a dimension. Indeed if one considers the two distributions (6.8) and (6.9) but

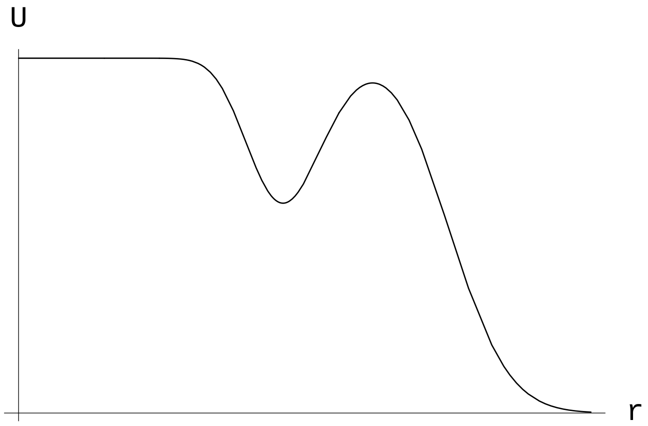

Figure 3: Distribution for a Schur polynomial in which the λ_p are equal. The figure shows $N = 120$, $n = 30$ with $\lambda_p = 2$ for $p \leq 15$.

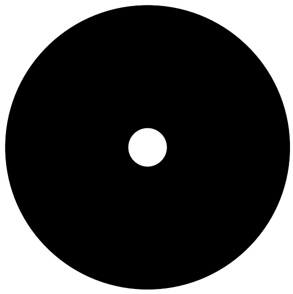

Figure 4: A maximal giant graviton corresponds to a disc with a small hole at the centre.

now with $n \geq N$, one finds that the vevs for the dimension four neutral operators are respectively

$$\langle \mathcal{O}_{S^{40}} \rangle = \frac{\sqrt{3}N}{\sqrt{5}\pi^2}(2\tilde{n} + 3\tilde{n}^2); \qquad \langle \mathcal{O}_{S^{40}} \rangle = \frac{\sqrt{3}N}{\sqrt{5}\pi^2}(2\tilde{n} + 3(\tilde{n}^2 - 2\tilde{n}_2\tilde{n}_1)), \tag{6.11}$$

where $\tilde{n} = n/N$, $\tilde{n}_1 = n_1/N$ and $\tilde{n}_2 = n_2/N$ with $\tilde{n} = \tilde{n}_1 + \tilde{n}_2 \geq 1$. The leading order terms as $N \rightarrow \infty$ for $(n_1, n_2) \gg 1$ have been retained. These vevs are indeed distinguishable even as $N \rightarrow \infty$; note that the vev for the single annulus is greater than that of the other distribution. Of course, following the general discussion given earlier, this distinguishability will still not be sufficient to determine the dual state $|\Phi\rangle$ uniquely: the single trace operator vevs cannot provide enough information. Moreover, again a typical Schur polynomial will not give rise to a distribution with strong peaks which is well approximated by annuli.

Consider a disc plus annulus configuration with fixed energy n. For given n one might think that the radius of the annulus $|w|^2 = 1 + \beta$ can be taken to be arbitrarily large provided that its width is decreased accordingly. However, one has to take into the flux quantization condition: as discussed in [5] one can integrate the five-form flux over a sphere surrounding the annulus, and the flux must be quantized. This quantization requires that the total area of the annulus (and indeed of any isolated droplet) must be a multiple of π/N. Thus the width of the annulus defined as $\delta = |w|^2_{\rm max} - |w|^2_{\rm min}$ is a multiple of $1/N$. The total energy of the configuration of the disc plus annulus, relative to the conformal vacuum, is given by $n = N\beta + 1$ and therefore for given n one gets $\beta = (n-1)/N$, as in (6.8). This is the maximal radius for a given energy: multiple annuli or thicker annuli necessarily have lower radii.

This same bound is also visible directly from the density distribution. First note that the function $e^{-\chi}\chi^\alpha/\alpha!$ for large α has support only around $\chi = \alpha$. Thus the distribution (6.1) extends to a maximum radius

$$|w|^2_{\rm max} = 1 + \frac{\lambda_1 - 1}{N}, \tag{6.12}$$

which is maximal for the Schur polynomial in which $\lambda_1 = n$ and $\lambda_i = 0$ otherwise. This implies an upper bound on the magnitudes of vevs of the scalar operators, for a given energy.

6.2 Non symmetric distributions

A distribution which is not radially symmetric corresponds to a superposition of R-eigenstates. The generic state $|\Phi\rangle$ can be written in terms of Schur polynomials as

$$|\Phi\rangle = \sum_{n,\lambda} a_{n,\lambda}\chi^n_{\{\lambda\}}|\Omega\rangle \equiv \sum_{n,\lambda} a_{n,\lambda}|n,\{\lambda\}\rangle. \tag{6.13}$$

with the Schur polynomials being orthonormal, and normalization of the state implies $\sum_{n,\lambda}|a_{n,\lambda}|^2 = 1$. Now consider an operator $\mathcal{O}_{\Delta,j}$ of dimension Δ, R-charge j. The vev of such an operator in this state is

$$\langle\mathcal{O}_{\Delta,j}\rangle_\Phi = \sum_{n,\lambda,\lambda'} a^*_{n+j,\lambda'}a_{n,\lambda}\langle(n+j),\{\lambda'\}|\mathcal{O}_{\Delta,j}|n,\{\lambda\}\rangle. \tag{6.14}$$

The three point functions appearing in this sum are non-extremal, except when $\Delta = j$, i.e. the operator is maximally charged. As discussed previously, the leading order contribution to non-extremal correlators as $N \to \infty$ is independent of the specific choices of Schur polynomial. Thus, the leading order contributions to the vevs of non-maximally charged operators can be computed using

$$|\Phi\rangle \approx \sum_n \tilde{a}_n|n\rangle, \tag{6.15}$$

where $|n\rangle$ is any state of R-charge n, and the coefficients $\tilde{a}_n$ are such that

$$\sum_{\{\lambda\}}|a_{n,\lambda}|^2 = |\tilde{a}_n|^2. \tag{6.16}$$

However, the vevs of the maximally charged operators do depend on the specific Schur polynomials appearing in the state $|\Phi\rangle$, because the corresponding three point functions

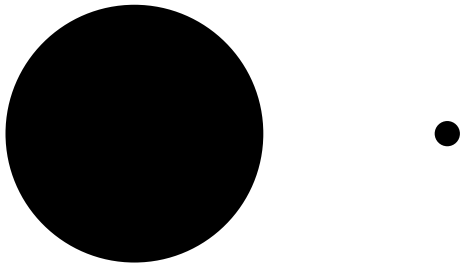

Figure 5: A disc plus droplet corresponds to a superposition of Schur polynomials of all dimension.

are extremal. Thus these vevs can be used to distinguish between different distributions with the same $\tilde{a}_n$, even at leading order in N. For example, consider the states

$$|\Phi\rangle = a_0|\Omega\rangle + a_{n,\{\lambda\}}|n,\lambda\rangle, \tag{6.17}$$

where $|a_0|^2 + |a_{n,\{\lambda\}}|^2 = 1$. To leading order in N, the vevs of the energy, R-charge and neutral operators are independent of the specific choice of $\{\lambda\}$. However, the vevs of the maximally charged operators with dimension n clearly do distinguish them. Let us compare the state created by the single trace operator, namely $|n,\lambda\rangle \to \mathcal{O}^{n,n}|\Omega\rangle$ where the single trace operator is defined in (5.5), and that associated with the Schur polynomial for which $\lambda_1 = n$, namely $|n,\lambda\rangle \to \hat{C}^{\dagger}_{N+n-1}\hat{C}_{N-1}|\Omega\rangle$. Then the expectation values of the single trace operator of charge $-n$ are respectively

$$\begin{aligned} |\Phi\rangle = a_0|\Omega\rangle + a_n\mathcal{O}^{n,n}|\Omega\rangle : \qquad & \langle\mathcal{O}^{n,-n}\rangle_\Phi = N_n a_0^* a_n e^{-int}; \\ |\Phi\rangle = a_0|\Omega\rangle + a_n\hat{C}^{\dagger}_{N+n-1}\hat{C}_{N-1}|\Omega\rangle : \qquad & \langle\mathcal{O}^{n,-n}\rangle_\Phi = \frac{N_n}{\sqrt{n}} a_0^* a_n e^{-int}, \end{aligned} \tag{6.18}$$

which differ by a factor of $\sqrt{n}$.

We should note here, however, that such superpositions which involve only a small number of Schur polynomials do not generically give rise to smooth supergravity solutions. In the example just given, taking (a_0, a_n) to be real, the corresponding distribution takes the form

$$\rho(w,\phi) = \rho(|w|) + \tilde{\rho}(|w|)\cos(n\phi), \tag{6.19}$$

with the functions $(\rho(|w|), \tilde{\rho}(|w|))$ dependent on the specific Schur polynomial. In the case that $|n,\lambda\rangle \to \hat{C}^{\dagger}_{N+n-1}\hat{C}_{N-1}|\Omega\rangle$ the functions are

$$\begin{aligned} \pi\rho(|w|) &= \theta(1-N^{-1}-|w|^2)+|w|^{2(N-1)}e^{-N|w|^2}\left(|a_0|^2\frac{N^{N-1}}{(N-1)!}+|a_n|^2\frac{N^{N+n-1}}{(N+n-1)!}|w|^{2n}\right); \\ \pi\tilde{\rho}(|w|) &= a_0 a_n |w|^{2(N-1)+n}e^{-N|w|^2}\frac{N^{N+n/2-1}}{\sqrt{(N-1)!(N+n-1)!}}. \end{aligned} \tag{6.20}$$

Regularity of the supergravity solution requires that $\pi\rho(w,\phi)$ takes the values $\{0,1\}$ everywhere. This condition can however never satisfied by a function of the form (6.19), with

$\tilde{\rho}(|w|)$ non-zero. One can see this easily as follows. Suppose the function $\rho(w,\phi)$ satisfies this condition at $\phi = 0$, so that

$$\pi\rho(w,0) = \rho(|w|) + \tilde{\rho}(|w|) = \{0,1\}, \tag{6.21}$$

for $\tilde{\rho}(|w|) \geq 0$. Then

$$\pi\rho(w,\delta\phi) - \pi\rho(w,0) = -\frac{1}{2}\pi n^2 \tilde{\rho}(|w|)\delta\phi^2 + \cdots \tag{6.22}$$

Regularity would require that the right hand side takes only the values $\{-1,0,1\}$ for all $|w|$ and $\delta\phi$, but this is not possible given that $\delta\phi^2$ is a continuous function of $\delta\phi$. Smoothing the distribution so that $\pi\rho(w,\phi)$ does take the values $\{0,1\}$ everywhere introduces additional terms (with small coefficients) into the state $|\Phi\rangle$, and such terms may not be distinguishable at leading order in N.

Conversely, a droplet distribution which is not radially symmetric but which gives rise to a regular supergravity solution is always associated with a superposition of an infinite number of Schur polynomials. Such a distribution can be written in terms of step functions whose arguments are the boundaries of the droplets:

$$\theta(|w|^2 - f(\phi)); \qquad f(\phi) = f_0 + \tilde{f}(\phi), \tag{6.23}$$

where f_0 is a zero mode whilst $\tilde{f}(\phi)$ has no zero mode and can be expanded in Fourier modes. Now even when the droplet boundary $\tilde{f}(\phi)$ contains only one frequency n, the distribution will contain all multiples of this frequency. One can see this by mode expanding the distribution, using (4.31), or by computing the multipole moments of the distribution as

$$\int d\phi d|w|^2 |w|^{2l} e^{im\phi}\rho = (1+l)^{-1} \int d\phi e^{im\phi}(f_0 + \tilde{f}(\phi))^{1+l}. \tag{6.24}$$

The latter integral is clearly non-zero when the frequency m is contained in $\tilde{f}(\phi)$ or its products, demonstrating that the distribution contains products of the frequencies contained in $\tilde{f}(\phi)$. Therefore, in the case that the droplet boundary contains only frequency n the corresponding state is a superposition of Schur polynomials of dimension kn involving all $k \geq 0$.

6.3 An example: a disc with a ripple

Consider a disc with a ripple such that the boundary of the distribution is at

$$|w|^2 = 1 + \alpha\cos(n\phi). \tag{6.25}$$

By the arguments above, such a distribution corresponds to a state $|\Phi\rangle$ which is a superposition of Schur polynomials of dimension kn, namely

$$|\Phi\rangle = \sum_{k,\{\lambda\}} a_{k,\{\lambda\}} |kn;\{\lambda\}\rangle. \tag{6.26}$$

In general, one will need to compute all one-point functions of all single and multitrace operators in order to deduce the coefficients $a_{k,\{\lambda\}}$. However, given that this distribution

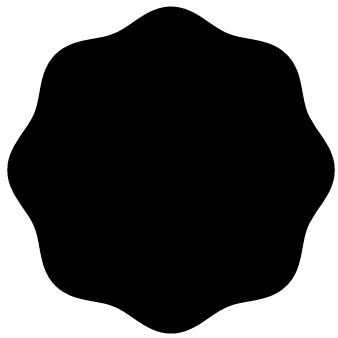

Figure 6: A disc with a ripple of frequency n corresponds to a superposition of Schur polynomials whose dimensions are multiples of n. The actual figure has $n = 8, \alpha = 0.1$.

is highly symmetric, with only one defining parameter, let us try to use the vevs of the lowest dimension single trace operators to deduce the superposition. Evaluating (3.56) for the distribution with boundary at (6.25), the energy and R-charge in this state are

$$E - E_c = J = \frac{1}{4} N^2 \alpha^2, \tag{6.27}$$

whilst from (3.59)–(3.60) the vevs of the neutral operators are

$$\langle \mathcal{O}_{S^{20}} \rangle = \frac{N^2 \sqrt{2}}{4\sqrt{3}\pi^2} \alpha^2; \quad \langle \mathcal{O}_{S^{40}} \rangle = \frac{N^2 \sqrt{3}}{2\sqrt{5}\pi^2} \alpha^2, \tag{6.28}$$

and from (3.58) the vevs of the maximally charged operators with dimension n are

$$\langle \mathcal{O}_{S^{n,\pm n}} \rangle = \frac{N^2}{2\pi^2 \sqrt{n}} (n-2) \sqrt{n-1} e^{\mp 2int} \alpha. \tag{6.29}$$

First note that the energy and the vevs of neutral operators are reproduced by a superposition of the form (6.15) with

$$|\Phi\rangle = e^{-\frac{N^2 \alpha^2}{4n}} \sum_{k=0}^{\infty} \frac{(N\alpha)^k}{(2\sqrt{n})^k \sqrt{k!}} |kn\rangle. \tag{6.30}$$

To compute the energy and the neutral vevs at leading order in N we may use any representative orthonormal state $|kn\rangle$, using the result of (6.4). Thus the energy is given by

$$\langle E - E_c \rangle_\Phi = e^{-\frac{N^2 \alpha^2}{2n}} \sum_{k=1}^{\infty} \frac{(N\alpha)^{2k}}{2^{2k} n^k k!} (kn) = \frac{1}{4} N^2 \alpha^2. \tag{6.31}$$

The vev of the dimension four operator can be computed using the result given in (B.6):

$$\begin{aligned} \langle \mathcal{O}^{40} \rangle_\Phi &= e^{-\frac{N^2 \alpha^2}{2n}} \sum_{k=0}^{\infty} \frac{(N\alpha)^{2k}}{2^{2k} n^k k!} \langle \mathcal{O}^{40} \rangle_{kn} \\ &= \frac{\sqrt{12} e^{-\frac{N^2 \alpha^2}{2n}}}{\sqrt{5}\pi^2} \sum_{k=1}^{\infty} \frac{(N\alpha)^{2k}}{2^{2k} n^k k!} (kn) = \frac{N^2 \sqrt{3}}{2\sqrt{5}\pi^2} \alpha^2. \end{aligned} \tag{6.32}$$

Now consider the vev of the maximally charged operator: as we have emphasized this vev is sensitive to the specific Schur polynomials contained in the state $|kn\rangle$. Let the single trace operator of dimension n be $\mathcal{O}^{n,n} = \mathcal{N}_n \hat{s}_n$, where by construction $\hat{s}_n|\Omega\rangle$ is orthonormal. Now suppose that $|kn\rangle$ is the state created by products of this operator such that

$$|kn\rangle = \frac{1}{\sqrt{k!}}(\hat{s}_n)^k|\Omega\rangle. \tag{6.33}$$

The states $|kn\rangle$ are orthonormal to each other in the large N limit. The easiest way to check this is to go back to SYM language where $|kn\rangle = \frac{1}{\sqrt{k!}}\left((\mathrm{Tr}Z^n)/\sqrt{N^n n}\right)^k|\Omega\rangle$ and then use standard large N counting. The leading contribution in $\langle kn|kn\rangle$ comes from disconnected diagrams where each $\mathrm{Tr}Z^n$ is contracted with a corresponding $\mathrm{Tr}\bar{Z}^n$.

Then by construction

$$\langle n(k+1)|\hat{s}_n|kn\rangle = \sqrt{k+1}, \tag{6.34}$$

which implies that

$$\begin{aligned} \langle \mathcal{O}_{S^{n,\pm n}}\rangle_\Phi &= \mathcal{N}_n e^{\mp 2int} e^{-\frac{N^2\alpha^2}{2n}} \sum_{k=0}^{\infty} \frac{(N\alpha)^{2k+1}}{2^{2k+1}n^k k!\sqrt{n(k+1)}}\langle n(k+1)|\hat{s}_n|kn\rangle \\ &= \frac{\mathcal{N}_n N\alpha}{2\sqrt{n}}e^{\mp 2int} = \frac{N^2}{2\pi^2\sqrt{n}}(n-2)\sqrt{n-1}e^{\mp 2int}\alpha, \end{aligned} \tag{6.35}$$

in exact agreement with the holographic result (6.29). By contrast, if $|kn\rangle$ were instead the state created by the single trace operator $\mathrm{Tr}(Z^{kn})$, then using the appropriate single trace operator three point functions one can show that the holographic result for the charged operator vev would not be reproduced: the vev would be down by a factor of N.

Note that when α is infinitesimal the state $|\Phi\rangle$ reduces to a perturbation of the conformal vacuum by the operator $\hat{s}_n|\Omega\rangle$:

$$|\Phi\rangle \to (1 + \frac{N\alpha}{2\sqrt{n}}\hat{s}_n)|\Omega\rangle, \tag{6.36}$$

in agreement with the identification of infinitesimal perturbations made in [9]. Thus the disc with a ripple is consistent with being dual to a state created by coherent superpositions of states created by powers of the single trace operators $\hat{s}_n$ acting on $|\Omega\rangle$. Indeed, the state $|\Phi\rangle$ can be written as a coherent state,

$$|\Phi\rangle = |\delta) \equiv e^{-\frac{1}{2}\delta^2}\sum_{k=0}^{\infty}\frac{\delta^k}{\sqrt{k!}}|kn\rangle, \tag{6.37}$$

for $\delta = N\alpha/2\sqrt{n}$ with $|kn\rangle$ the state containing k quanta of $\hat{s}_n$.

6.4 Ripplon deformations and the chiral boson description

The identification of the ripple deformed disc with a state built from a coherent superposition of the operators $\hat{s}_n$ follows naturally from earlier discussions of edge excitations in quantum Hall liquids, see for example [39, 40]. Consider a distribution consisting of a single droplet, whose boundary can be parametrized as

$$|w|^2 = 1 + X(\phi), \tag{6.38}$$

where $X(\phi)$ is an arbitrary function with no zero modes. This function $X(\phi)$ describes area preserving deformations of the disc. Now so far we have used the fermion picture, that is, the Schur polynomials, to describe excitations relative to the conformal vacuum. This is a natural basis to describe droplets which are separated from the disc, but it is not the natural basis for describing coherent ripplon deformations in the shape of the droplet.

Such ripplons can best be described by quantizing the chiral boson field $X(\phi)$; quantizing its Fourier mode expansion gives rise to a Hilbert space associated with bosonic creation and annihilation operators. As discussed in [40], these operators can in turn be identified with elements in the symmetric polynomial (or equivalently, the trace) basis. Thus edge waves and deformations in the shape of the droplet are most naturally described within the trace (chiral boson) description rather than the Schur polynomial (fermion) description. This extends the identification made for the single frequency ripple discussed above to more general ripples.

It is interesting to note that the algebra of area preserving diffeomorphisms of the droplet is actually W_∞. It emerges in both the fermionic and bosonic formulations as the algebra of unitary transformations of physical states. It would be interesting to understand the meaning and implications of W_∞ for holography.

7. Discussion

In this paper we have discussed holography for bubbling solutions. Solutions that are asymptotically $AdS_p \times S^q$ contain an infinite amount of holographic data that can be extracted by *algebraic manipulations*, namely the holographic 1-point functions that characterize the vacuum of the dual QFT. This is the simplest information one can extract from a given supergravity solution. Conversely, knowledge of the 1-point functions is in principle sufficient in order to reconstruct bulk solutions from QFT data. Two-point and higher-point functions can also in principle be extracted, but this requires solving at least the linearized equations around the solution (for 2-point functions) whilst for n-point functions one needs to solve the $(n-1)$-th order equations. Explicitly solving such equations is an intractable problem, except for very symmetric solutions. Moreover, in the case of interest vevs are protected, given the non-renormalization of 3-point functions of chiral primaries at the conformal vacuum, whilst there are no such non-renormalization theorems protecting generic two point functions, and corresponding four point functions in the conformal vacuum.

In the first part of this paper we reviewed the holographic 1-point functions derived in [4] for the stress energy tensor, the R-currents and all chiral primaries up to dimension four for asymptotically $AdS_5 \times S^5$ solutions of IIB supergravity that involve the metric and the 5-form. These results hold generally when the solution is dual to a state (rather than describing a deformation), i.e. they do not dependent on the amount of supersymmetry preserved by the solution or its bosonic isometries (except that the solution should be asymptotic to $AdS_5 \times S^5$), and thus these 1-point functions can be used to extract holographic data from any such solution. The 1-point functions are given in terms of the asymptotic coefficients of the ten dimensional solution and are presented in (2.28), (2.34) (2.35). Note that the expressions are non-linear in the asymptotic coefficients.

The next step was the evaluation of these holographic formulas on the LLM solutions. The holographic 1-point functions are by construction diffeomorphism covariant, so the asymptotic coefficients can be extracted in any coordinate system. A clever choice of coordinates however can reduce the required labor significantly. Recall that the LLM solution is determined by a harmonic function Φ in six dimensions with sources on a 2-plane. The asymptotic expansion around $AdS_5 \times S^5$ can be efficiently performed by writing $\Phi = \Phi^o + \Delta\Phi$, where Φ^o is the harmonic function that leads to $AdS_5 \times S^5$, and then expanding in $\Delta\Phi$. One still needs to convert these expansions into radial expansions. This is done by first using flat coordinates on R^6 so that the asymptotic expansion of $\Delta\Phi$ takes a standard form and then transforming to the coordinates most natural for the $AdS_5 \times S^5$ solution. This procedure minimizes the number of non-linear terms entering the computation of the 1-point functions.

We obtained explicit expressions for the vevs in terms of integrals over the 2-plane defining the solution; these are given in (3.54)–(3.56)–(3.58)–(3.59)–(3.60). Note that in (3.58) we give the vevs of all maximally charged operators, i.e. operators with dimension equal to R-charge, despite the fact that the general analysis in the previous section was done for operators up to dimension four. The ability to produce such a result is due to special properties of the LLM solution combined with the previous Coulomb branch results of [2]. Given the large amount of supersymmetry preserved by the LLM solutions, one would expect that these vevs should not renormalize and thus that they must be reproduced by a weak coupling computation. Put differently, these vevs provide checks for the correct identification of the dual theory.

The vevs satisfy a number of non-trivial consistency checks. Firstly, the vev of the energy is proportional to the vev of the R-charge (up to the the Casimir energy of SYM on S^3) as is required by supersymmetry. Secondly, all vevs, except for the energy which should become equal to the Casimir energy, should vanish for the theory at the conformal vacuum and this is indeed the case for the vevs we derive. Thirdly, the LLM solutions in the decompactification limit of S^3 go over to SO(4) symmetric distributions of D3 branes, with the sources on the 2-plane now describing the distribution of D3 branes. These solutions are dual to $N = 4$ SYM on the Coulomb branch. The LLM vevs indeed correctly reduce to the Coulomb branch vevs given in [2] in this limit.

Before proceeding we should make a comment about the mass of these spacetimes.[6] Had the solutions been asymptotically flat, one would have obtained their mass from the g_{tt} component of the metric. In our case however the solution is asymptotically $AdS_5 \times S^5$ so such a prescription is in general not valid. There are two issues here. One is that the

[6] An earlier discussion about the mass of the LLM solutions can be found in [41].

solution involves in a non-trivial fashion a compact part of the geometry and the other is that the non-compact part is asymptotically AdS. For asymptotically AdS spacetimes the issue of mass has been revisited and thoroughly analyzed in recent years [23 − 25, 44, 45] resulting in holographic formulas which relate the mass to the asymptotics of the metric and other matter fields.

Taking into account the compact part is also non-trivial since none of the existing consistent truncation formulas from ten to five dimensions is directly applicable. One can however reduce the solution to five dimensions without truncating, keeping all fields relevant for the computation of the mass. This is essentially the method of KK holography [4] and results in the rigorous formula for the holographic stress energy tensor given in (2.34). One can then obtain the mass from the T_{tt} component, as usual.

The field theory dual to the LLM solutions is expected to be $N = 4$ SYM on $R \times S^3$ in a half supersymmetric state. A general way of analyzing this theory would be to carry out path integral quantization. The requisite supersymmetry is preserved by quantizing around 1/2 supersymmetric solutions of $N = 4$ SYM on $R \times S^3$. Examples of such solutions were discussed in [42] and more recently in [43]. These solutions are time-dependent and in correspondence with the Coulomb branch of $N = 4$ SYM on $R^{(3,1)}$. In particular, the curvature coupling implies the scalar Z satisfies an equation of the form $\dot{Z} \sim iZ$. This implies that the R-symmetry current $j_\mu \sim \mathrm{Tr}\bar{Z} \overleftrightarrow{\partial_\mu} Z$ and the operator $\mathcal{O}^{2,0} \sim \mathrm{Tr}\bar{Z}Z$ are proportional to the each other when evaluated on these solutions (and similarly for related higher dimension operators). This provides an additional consistency check for the holographic vevs, which the vevs in (3.54)–(3.59) indeed satisfy.

Due to the extended supersymmetry one might expect that the exact values of the vevs of 1/2 supersymmetric gauge invariant operators could be computed by a semiclassical computation. This would provide a rigorous computation of the vevs from first principles. Actually carrying out this computation is not so easy in practice, though, because of subtleties associated with correctly treating the integration measure. The issue is the following: for the computations of interest one will want to integrate out most of the SYM fields, including the other four scalars X^i and off-diagonal degrees of freedom of the complex matrix Z, leaving only an integral over the eigenvalues of this matrix. This in turn involves correctly parametrizing the path integral measure as given in

$$\mathcal{Z} = \int [dZ dZ^\dagger \prod_i dX^i \cdots] e^{iS_{\mathrm{YM}}[Z, Z^\dagger, X^i, \ldots]}, \tag{7.1}$$

and then integrating out appropriately. Now for a general computation one does not expect to be able to integrate out exactly all these degrees of freedom. Integrating first over the S^3 would lead to a complicated interacting multi-matrix model which will not in general be solvable. However, the holographic computations for the vevs along with the fact that we can reproduce them by the holomorphic matrix model, imply that at least for these computations one *can* explicitly integrate out the other degrees of freedom. Demonstrating this by a first principles computation would be useful as it would explain the regime of validity and the limitations of the free fermion description. Moreover, one may in this way show how certain computations can be carried out in multi-matrix models, even when they are not exactly solvable.

In the absence of a rigorous derivation of the free fermion description from first principles, we proceeded by using it as a working assumption. On symmetry grounds the state that any given bubbling solution is dual to is a superposition of states obtained from the conformal vacuum by the action of a 1/2 BPS operator. The question is then whether one can uniquely determine the precise superposition from the data encoded in the solution. Using the identification of the coloring of the 2-plane with the phase space distribution of the free fermions we show that this information alone does not completely determine the state in the large N limit. It does determine it enough however so that the vevs of all single trace 1/2 BPS operators in that state are uniquely determined. This is precisely the information encoded holographically in the asymptotics of the solution. The missing information is related to vevs of multi-trace operators.

A general single trace 1/2 BPS operator depends on fields other than the complex Z field. This implies that these operators cannot not in general be implemented with free fermions. Nevertheless, we showed that for the purpose of the computation of the vevs and to leading order in N such an implementation is possible and all such operators are expressed in terms of bilinears of fermion creation and annihilation operators. Using these expressions we show that *all* vevs computed holographically agree *exactly* with the field theory computation in the large N limit for any distribution.

To illustrate our discussion we analyzed a number of examples. In accordance with our general discussion, we showed that all vevs associated with any symmetric distributions are degenerate to leading order N. For non-symmetric distributions, the vevs of charged operators (which by symmetry considerations are zero in symmetric distributions) can (partly) distinguish between different states. However, an infinite superposition of states of definite R-charges is required to obtain a regular geometry. We also analyzed in detail the case of the distribution being a ripple on a disc. This case has been analyzed previously for an infinitesimal ripple in [9]. We showed here that a finite ripple corresponds to a coherent state of single trace operators.

We should also comment on the striking parallels between this system and the 2-charge D1-D5 fuzzball solutions. Both systems can be characterized by a set of curves: in the LLM case these are curves in R^2 describing the droplet boundaries, whilst in the D1-D5 case these are curves in an auxiliary space describing the supertube shape and its internal excitations. The holographic analysis for this system has recently been done in [46]. In both cases, only when the curves are circular and preserve rotational symmetry do the geometries correspond to vacua built from a single operator (in the R-charge basis). Regular geometries in which the rotational symmetry is broken correspond to infinite superpositions of states in the R-charge basis, with the coefficients of the superpositions related to the Fourier expansions of the curves. Thus the natural bases in the dual field theory, which are labeled by their R charges, are not the natural bases for regular geometries.

It would be interesting to use the holographic anatomy techniques discussed in this paper to analyze 1/4 and 1/8 BPS bubbling solutions [34]. One would expect that these include both geometries dual to states and those dual to deformations. A holographic

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analysis should determine how the boundary conditions which ensure regularity in the interior of these geometries are related to the vevs/deformations in the dual theory. More generally one may hope that combining supersymmetric classification techniques with holographic anatomy might lead to more efficient holographic engineering of geometries dual to supersymmetric field theory states and deformations.

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A. Properties of spherical harmonics

The defining equations for the spherical harmonics are

$$\begin{aligned}
\Box_y Y^{I_1} &= \Lambda^{I_1} Y^{I_1}, & \Lambda^{I_1} &= -k(k+4), & k &= 0,1,2,\dots \\
\Box_y Y_a^{I_5} &= \Lambda^{I_5} Y_a^{I_5}, & \Lambda^{I_5} &= -(k^2+4k-1), & k &= 1,2,\dots \\
\Box_y Y_{(ab)}^{I_{14}} &= \Lambda^{I_{14}} Y_{(ab)}^{I_{14}}, & \Lambda^{I_{14}} &= -(k^2+4k-2), & k &= 2,3,\dots \\
\Box_y Y_{[ab]}^{I_{10}} &\equiv \Lambda^{I_{10}} Y_{[ab]}^{I_{10}}, & \Lambda^{I_{10}} &= -(k^2+4k-2), & k &= 1,2,\dots \\
D^a Y_a^{I_5} &= D^a Y_{(ab)}^{I_{14}} = D^a Y_{[ab]}^{I_{10}} = 0.
\end{aligned} \tag{A.1}$$

The overall normalization is chosen so that the harmonics are normalized as

$$\int Y^{I_1} Y^{I_2} = \pi^3 z(k) \delta^{I_1 I_2}, \qquad z(k) = \frac{1}{2^{k-1}(k+1)(k+2)} \tag{A.2}$$

The triple overlap between spherical harmonics is defined as

$$\int Y^{I_1} Y^{I_2} Y^{I_3} = \pi^3 a_{I_1 I_2 I_3}. \tag{A.3}$$

Recall that the scalar harmonics can be represented as

$$Y^{I_1} = C^{I_1}_{i_1 \cdots i_k} x^{i_1} \cdots x^{i_k} \tag{A.4}$$

where x^{i_n} are Cartesian coordinates on S^5 and $C^I_{i_1\cdots i_k}$ is a totally symmetric traceless rank k tensor of SO(6). The normalization in (A.2) corresponds to delta function normalization for the C^I's, i.e.

$$\langle C^{I_1} C^{I_2} \rangle \equiv C^{I_1}_{i_1 \cdots i_k} C^{I_2 i_1 \cdots i_k} = \delta^{I_1 I_2}. \tag{A.5}$$

Note that

$$a_{I_1 I_2 I_3} = \frac{1}{(\frac{1}{2}\Sigma + 2)! 2^{\frac{1}{2}(\Sigma - 2)}} \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!} \langle C^{I_1} C^{I_2} C^{I_3} \rangle. \tag{A.6}$$

where $\Sigma = k_1 + k_2 + k_3$, $\alpha_1 = \frac{1}{2}(k_2 + k_3 - k_1)$ etc. Useful identities for the scalar harmonics include

$$\begin{aligned} D^a D_{(a} D_{b)} Y^I &= 4\left(1 + \frac{\Lambda^I}{5}\right) D_a Y^I; \\ \Box_y D_{(a} D_{b)} Y^I &= (10 + \Lambda^I) D_{(a} D_{b)} Y^I; \\ \Box_y D_a Y^I &= (\Lambda^I + 4) D_a Y^I. \end{aligned} \tag{A.7}$$

Vector harmonics are normalized so that

$$\int Y_a^{I_1} Y^{I_2 a} = z(k) \delta^{I_1 I_2}, \tag{A.8}$$

where $z(k)$ is as given in (A.2). We introduce the following coordinates on S^5

$$ds^2 = d\theta^2 + \cos^2\theta d\Omega_3^2 + \sin^2\theta d\phi^2. \tag{A.9}$$

The differential equation (A.1) for the scalar harmonics is separable. Imposing SO(4) symmetry implies that the spherical harmonics depend only on θ and ϕ. The general solution can then be expressed in terms of a hypergeometric functions,

$$Y^{(k,m)}(\theta, \phi) = c_{(n,m)} y_m^k(\theta) e^{im\phi} \tag{A.10}$$

where $c_{(n,m)}$ is a normalization constant and the function $y_m^k(\theta)$ is given by

$$y_m^k(x) = x^{|m|} {}_1F_2\left(-\frac{1}{2}(k - |m|), 2 + \frac{1}{2}(k + |m|), 1 + |m|; x^2\right) \tag{A.11}$$

with $x = \sin\theta$ (there are also a second solution with leading behavior $x^{-|m|}$ but this solution does not reduces to a finite polynomial for any choice of the quantum numbers). The hypergeometric function reduces to a finite polynomial when either the first or second argument is zero or a negative integer. This leads to the following cases

$$(k = 2l, \quad m = 2n), \qquad (k = 2l + 1, \quad m = 2n + 1) \qquad n \in [-l, l], \ l \in Z^+ \tag{A.12}$$

with

$$\begin{aligned} y_{2n}^{2l}(x) &= x^{2|n|} {}_1F_2(-l + |n|, 2 + l + |n|, 1 + 2|n|; x^2) \\ y_{2n+1}^{2l+1}(x) &= x^{|2n+1|} {}_1F_2(-l + |n|, 3 + l + |n|, 2 + 2|n|; x^2) \end{aligned} \tag{A.13}$$

The harmonics that are also SO(2) symmetric are given by

$$Y^{(2l,0)}(\theta, \phi) = \frac{(-)^l}{2^l \sqrt{2l + 1}} \left(\sum_{m=0}^{l} (-)^m \begin{pmatrix} l \\ m \end{pmatrix} \begin{pmatrix} l + m + 1 \\ l + 1 \end{pmatrix} (\sin\theta)^{2m} \right). \tag{A.14}$$

The lowest harmonics are therefore

$$\begin{aligned} Y^{(2,0)} &= \frac{1}{2\sqrt{3}}(3\sin^2\theta - 1), \\ Y^{(4,0)} &= \frac{1}{4\sqrt{5}}(10\sin^4\theta - 8\sin^2\theta + 1), \end{aligned} \tag{A.15}$$

We will also need the following normalized charged scalar harmonics

$$Y^{(k,\pm k)} = \frac{1}{2^{k/2}} \sin^k \theta e^{\pm ik\phi}; \tag{A.16}$$

$$Y^{(3,\pm 1)} = \frac{\sqrt{3}}{4} \sin\theta (2\sin^2\theta - 1) e^{\pm i\phi}; \tag{A.17}$$

$$Y^{(4,\pm 2)} = \frac{1}{2\sqrt{10}} \sin^2\theta (5\sin^2\theta - 3) e^{\pm 2i\phi}; \tag{A.18}$$

Note that the triple overlap between charged and neutral harmonics is given by

$$\langle C^{(k,-k)} C^{(k,k)} C^{(2p,0)} \rangle = \frac{1}{2^{p-1}\sqrt{2p+1}}, \tag{A.19}$$

where $C^{(p,q)}$ denotes the symmetric tensor corresponding to the degree p, SO(2) charge q spherical harmonic. The relevant vector harmonics are those with only components along ϕ:

$$Y^1 = \frac{1}{\sqrt{2}} \sin^2\theta d\phi; \tag{A.20}$$

$$Y^3 = \frac{\sqrt{3}}{2} \sin^2\theta (2\sin^2\theta - 1) d\phi. \tag{A.21}$$

B. Scalar chiral primaries

The single trace scalar chiral primary operators of dimension k are defined as

$$\mathcal{O}_{S^{kI}} = \frac{\mathcal{N}_k}{N^{k/2}\sqrt{k}} C^I_{i_1\cdots i_k} \mathrm{Tr}(X^{m_{i_1}} \cdots X^{m_{i_k}}) \tag{B.1}$$

where the properties of the degree k symmetric traceless tensors $C^I_{I_1\cdots i_k}$ are given in appendix A. The operators are normalized such that

$$\langle \mathcal{O}_{S^{k_1 I_1}}(x) \mathcal{O}_{S^{k_2 I_2}}(y) \rangle = \mathcal{N}^2_{k_1} \delta^{k_1 k_2} \frac{\delta^{I_1 I_2}}{|x-y|^{2k_1}}, \tag{B.2}$$

where the scalar fields X^m are normalized such that

$$\langle X^m_a(x) X^n_b(y) \rangle = \frac{\delta_{ab}\delta^{mn}}{|x-y|^2}, \tag{B.3}$$

where (a,b) are color indices. The appropriate normalization of the dimension k chiral primaries to match with supergravity is

$$\mathcal{N}^2_k = \frac{N^2}{\pi^4}(k-1)(k-2)^2, \tag{B.4}$$

for $k \neq 2$ with $\mathcal{N}^2_2 = N^2/\pi^4$.

The planar three point function for such scalar chiral primaries is given by

$$\langle \mathcal{O}_{S^{k_1 I_1}}(x) \mathcal{O}_{S^{k_2 I_2}}(y) \mathcal{O}_{S^{k_3 I_3}}(z) \rangle = \frac{\mathcal{N}_{I_1}\mathcal{N}_{I_2}\mathcal{N}_{I_3}}{N} \frac{\sqrt{k_1 k_2 k_3} \langle C^{I_1} C^{I_2} C^{I_3} \rangle}{|x-y|^{2\alpha_3}|y-z|^{2\alpha_1}|x-z|^{2\alpha_2}}. \tag{B.5}$$

Here $2\alpha_3 = k_1 + k_2 - k_3$ and (α_1, α_2) are defined analogously. The triple overlap of the symmetric traceless tensors is denoted $\langle C^{I_1} C^{I_2} C^{I_3} \rangle$; recall that these tensors are orthonormal (A.5).

Now let us consider the specific case of three point functions between one neutral (SO(4)$\times$SO(2) singlet) operator $\mathcal{O}^{2k,0}$ with dimension $2k$ and two conjugate SO(2) charged operators $\mathcal{O}^{n,n}$. The corresponding spherical harmonics are given in (A.14) and (A.16) respectively, with the triple overlap being given in (A.19). The three point function implies that the vev of the neutral operator in the (unit normalized) state created by $\mathcal{O}^{n,n}$ is

$$\langle \mathcal{O}^{2k,0} \rangle = \mathcal{N}_{2k} \frac{n\sqrt{2k}}{2^{k-1} N \sqrt{2k+1}}, \tag{B.6}$$

Therefore the vevs of the neutral operators in these states are given by

$$\langle \mathcal{O}^{2,0} \rangle_n = \frac{\sqrt{2}n}{\pi^2 \sqrt{3}}; \qquad \langle \mathcal{O}^{2k,0} \rangle_n = \frac{n}{\pi^2} \frac{(k-1)}{2^{k-2}} \sqrt{\frac{2k(2k-1)}{2k+1}}. \tag{B.7}$$

We will make use of several other three point functions, involving two maximally charged operators:

$$\begin{aligned} \langle \mathcal{O}^{3,-3} \mathcal{O}^{3,1} \mathcal{O}^{2,2} \rangle &= \frac{\mathcal{N}_3}{N} 3\sqrt{2} \langle C^{(3,-3)} C^{(3,1)} C^{(2,2)} \rangle = \sqrt{3} \frac{\mathcal{N}_3}{N}; \\ \langle \mathcal{O}^{4,-4} \mathcal{O}^{4,2} \mathcal{O}^{2,2} \rangle &= \frac{\mathcal{N}_4}{N} 4\sqrt{2} \langle C^{(4,-4)} C^{(4,2)} C^{(2,2)} \rangle = 4 \frac{\mathcal{N}_4}{\sqrt{5} N}, \end{aligned} \tag{B.8}$$

where $C^{(p,q)}$ denotes the symmetric tensor corresponding to the degree p, SO(2) charge q spherical harmonic.

C. Large N behavior of three point functions

To compute vevs of single trace chiral primary operators in generic half BPS states we use the corresponding three point functions. To determine the dominant effects in the large N limit we thus need to know the N dependence of correlators of the form

$$C_{\sigma_n^I} C_{\sigma_m^J} \langle (\mathrm{Tr}(\sigma_n^I Z))^* \mathcal{O}^{\mathcal{A}} \mathrm{Tr}(\sigma_m^J Z) \rangle \tag{C.1}$$

where $\mathcal{O}^{\mathcal{A}}$ is a single trace chiral primary, σ_n^I denotes a conjugacy class of S_n and the normalization factors $C_{\sigma_n^I}$ are such that the operators are orthonormal in the large N limit:

$$C_{\sigma_n^I} C_{\sigma_m^J} \langle (\mathrm{Tr}(\sigma_n^I Z))^* \mathrm{Tr}(\sigma_m^J Z) \rangle = \delta_{nm} \delta^{IJ} + \mathcal{O}(1/N). \tag{C.2}$$

Using the propagators given in (B.3) one finds that $C^2_{\sigma_n^I} \sim 1/N^n$; note that throughout this section we will suppress factors of order one. It is convenient to introduce the notation

$$\mathcal{O}_n^{\sigma[m]} = \frac{1}{N^{n/2}} \prod_i \mathrm{Tr}(Z^{n_i}); \qquad \sum_i n_i = n; \qquad \sum_i 1 = m, \tag{C.3}$$

for an operator of dimension n involving m traces with a permutation labeled by $\sigma[m]$. As discussed in [36] there are two distinct cases of correlators to consider, the extremal correlators in which the dimension of the conjugate operator is equal to the sum of the dimensions of the other operators and non-extremal correlators.

Let us consider first non-extremal correlators, focusing on the case where $\mathcal{O}^{\mathcal{A}}$ is an SO(2) neutral operator, namely it is an operator $\mathcal{O}^{2p,0}$ of dimension $2p$ such that

$$\mathcal{O}^{2p,0} = \frac{1}{N^p}\mathrm{Tr}(\bar{Z}^p Z^p + \cdots), \tag{C.4}$$

where the ellipses denote cyclic permutations. Now charge conservation implies that the correlator

$$\langle (\mathcal{O}_{n_1}^{\sigma[m_1]})^\dagger \mathcal{O}^{2p,0} (\mathcal{O}_{n_2}^{\sigma[m_2]}) \rangle \tag{C.5}$$

is only non-zero when $n_1 = n_2$. The N dependence varies according to the specific choices of $(\sigma[m_1], \sigma[m_2])$. As discussed in [36], for a generic choice the correlator will have the same N scaling as the related $(m_1 + m_2 + 1)$-point correlator of single trace operators, namely as $1/N^{m_1+m_2-1}$; thus for single trace operators the scaling is $1/N$. Recall that an n-point correlator of single trace operators behaves as

$$\langle \mathcal{O}^{k_1}\mathcal{O}^{k_2}\cdots\mathcal{O}^{k_n}\rangle \sim \frac{1}{N^{n-2}}, \qquad n \geq 2. \tag{C.6}$$

However, for specific choices of $(\sigma[m_1], \sigma[m_2])$ the N scaling can be enhanced, because there are disconnected components to the diagrams. In particular, large N counting gives

$$\langle (\mathcal{O}_{n}^{\sigma[m]})^\dagger \mathcal{O}^{2p,0} (\mathcal{O}_{n}^{\sigma[m]}) \rangle \sim \frac{1}{N}, \tag{C.7}$$

for any m, whilst for $\sigma[m_1] \neq \sigma[m_2]$ one always finds a subleading N dependence, with the 3-point function being at most of order

$$\langle (\mathcal{O}_{n}^{\sigma[m_1]})^\dagger \mathcal{O}^{2p,0} (\mathcal{O}_{n}^{\sigma[m_2]}) \rangle \sim \frac{1}{N^2}. \tag{C.8}$$

(This result for $m_1 = 1$ was given in [36].) Thus vevs of neutral operators are thus dominated by diagonal three point functions of the type (C.7). For the vevs of non-maximally charged operators, the relevant correlators are also non-extremal; the leading terms scale as $1/N$ and arise from single trace correlators and specific multi-trace correlators. We will not however need detailed results for the latter.

Now let us turn to the extremal correlators in which $\mathcal{O}^{\mathcal{A}}$ is a maximally charged single trace operator. Again the correlator involving single trace operators behaves as $1/N$, but in this case correlators involving multi trace operators can dominate, since they can grow as 1. In particular,

$$\langle (\mathcal{O}_{n+k}^{\sigma[m+1]})^\dagger \mathcal{O}^{k,k} (\mathcal{O}_{n}^{\sigma[m+1]}) \rangle \sim 1, \qquad \mathcal{O}_{n+k}^{\sigma[m+1]} = \mathcal{O}^{k,k}(\mathcal{O}_{n}^{\sigma[m]}). \tag{C.9}$$

Note that analogous results are obtained in the Schur polynomial basis; see [6] for related discussions.

D. Killing spinors for LLM solutions

We discuss in this appendix the computation of the Killing spinors of the LLM solutions. This computation was carried out in appendix A of [5] but only half of Killing spinors were correctly identified, even though the projection operators were given correctly, and furthermore the spacetime dependence is not given correctly. These corrections do not affect the final answer for the supergravity solution (although some intermediate steps in the derivation are affected). They may have a real effect however in similar computations for less supersymmetric solutions. Furthermore the correct Killing spinors may be needed for other purposes, for example for analyzing supersymmetric probe branes in this background.

We use the notation of [5] and choose the same basis of gamma matrices

$$\Gamma_\mu = \gamma_\mu \otimes 1 \otimes 1 \otimes 1, \quad \Gamma_a = \gamma_5 \otimes \sigma_a \otimes 1 \otimes \hat{\sigma}_1, \quad \Gamma_{\tilde{a}} = \gamma_5 \otimes 1 \otimes \tilde{\sigma}_a \otimes \hat{\sigma}_2, \tag{D.1}$$

where $\sigma_a, \tilde{\sigma}_a, \hat{\sigma}_a$ are the Pauli matrices.

The ten dimensional spinor is decomposed as

$$\eta = \epsilon_a \otimes \chi_a \otimes \tilde{\chi}_a \tag{D.2}$$

where $\chi_a, \tilde{\chi}_a$ are geometric Killing spinors of S^3, i.e. they obey

$$\nabla_c \chi_a = a\frac{i}{2}\gamma_c \chi_a, \qquad a = \pm 1, \tag{D.3}$$

and a similar equation for $\tilde{\chi}_a$, where ∇_c is the standard connection on a unit 3-sphere. We normalize these spinors as $\chi_a^\dagger \chi_a = \tilde{\chi}_a^\dagger \tilde{\chi}_a = 1$. The fact that the spinors are correlated as in (D.2) follows from the analysis in [5].

The Killing spinor equation then reduces to [5]

$$(iae^{-\frac{1}{2}(H+G)}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\gamma^\mu\partial_\mu(H+G))\epsilon + 2M\epsilon = 0, \tag{D.4}$$

$$(iae^{-\frac{1}{2}(H-G)}\gamma_5\hat{\sigma}_2 + \frac{1}{2}\gamma^\mu\partial_\mu(H-G))\epsilon - 2M\epsilon = 0, \tag{D.5}$$

$$\nabla_\mu\epsilon + M\gamma_\mu\epsilon = 0 \tag{D.6}$$

where

$$M = -\frac{1}{4}e^{-\frac{3}{2}(H+G)}\gamma^{\mu\nu}F_{\mu\nu}\gamma^5\hat{\sigma}_1 \tag{D.7}$$

Processing these equations one finds that the spinor ϵ_a should satisfy the following equations [5]

$$P_a^- \epsilon_a = R_a^+ \epsilon_a = 0, \tag{D.8}$$

where we introduce the commuting projection operators

$$P_a^\pm = \frac{1}{2}\left(1 \pm (ie^{-G}\gamma_5 + a\sqrt{1+e^{-2G}}\Gamma_3\hat{\sigma}_1)\right), \qquad R_a^\pm = \frac{1}{2}(1 \pm ia\Gamma_1\Gamma_2) \tag{D.9}$$

satisfying

$$(P_a^\pm)^2 = P_a^\pm, \quad P_a^+P_a^- = 0, \quad (R_a^\pm)^2 = R_a^\pm, \quad R_a^+R_a^- = 0 \quad [P_a^\pm, R_a^\pm] = 0 \tag{D.10}$$

Each of this projection cuts the number of spinors by 1/2, so we have a total of 8 Killing spinors for $a = +1$ and 8 Killing spinors for $a = -1$. The most general solution of (D.8) is

$$\epsilon_a = R_a^- P_a^+ \tilde{\epsilon}_a \tag{D.11}$$

where $\tilde{\epsilon}_a$ are (at this point) unconstrained spinors.

In [5] the following solution of (D.8) was given,

$$\epsilon = e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1}\epsilon_1, \qquad \Gamma^3\hat{\sigma}^1\epsilon_1 = a\epsilon_1, \qquad \sinh 2\delta = ae^{-G} \tag{D.12}$$

These are in fact only half of the Killing spinors in (D.11). To see this introduce a new projector,

$$S_a^\pm = \frac{1}{2}(1 \pm a\Gamma^3\hat{\sigma}^1), \qquad (S_a^\pm)^2 = S_a^\pm, \quad S_a^+ S_a^- = 0, \tag{D.13}$$

and decompose $\tilde{\epsilon}_a$ as

$$\tilde{\epsilon}_a = \tilde{\epsilon}_a^+ + \tilde{\epsilon}_a^-, \qquad S_a^\pm \tilde{\epsilon}_a^\pm = \tilde{\epsilon}_a^\pm, \quad S_a^\pm \tilde{\epsilon}_a^\mp = 0 \tag{D.14}$$

A short computation yields,

$$P_a^+ \tilde{\epsilon}_a^+ = \cosh\delta e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1}\tilde{\epsilon}_a^+ \tag{D.15}$$

which is the spinor in (D.12). Upon multiplication by R_a^- one obtains half of the Killing spinors in (D.11), namely we miss the ones based on $\tilde{\epsilon}_a^-$.

To specify the Killing spinor we need to specify $\tilde{\epsilon}_a$. To this end we consider the fermion bilinear $f_2 = i\bar{\epsilon}\hat{\sigma}_2\epsilon$. It was shown in [5] that f_2 equals

$$f_2 = e^{\frac{1}{2}(H+G)} \tag{D.16}$$

Inserting the spinors in (D.11) and defining

$$\tilde{\epsilon}_a^\pm = c^\pm e_a^\pm \tag{D.17}$$

we find that (D.16) implies

$$c^\pm = \frac{e^{\frac{1}{4}(H+G)}}{\sqrt{\sqrt{1+e^{-2G}} \pm 1}} \tag{D.18}$$

and

$$i\bar{e}_a\hat{\sigma}_2 R_a^- e_a = 2 \tag{D.19}$$

where

$$e_a = e_a^+ + i\gamma_5 e_a^-, \tag{D.20}$$

The Killing spinor becomes

$$\begin{aligned} \epsilon_a &= \frac{1}{\sqrt{2}} e^{\frac{1}{4}(H+G)} R_a^- \left((\cosh\delta + ia\gamma_5 \sinh\delta)e_a^+ + (-\sinh\delta + ia\gamma_5\cosh\delta)e_a^-\right) \\ &= \frac{1}{\sqrt{2}} e^{\frac{1}{4}(H+G)} R_a^- \left(e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1} e_a^+ + ia\gamma_5 e^{-i\delta\gamma^5\Gamma^3\hat{\sigma}^1} e_a^-\right) \end{aligned} \tag{D.21}$$

From these spinors one can construct appropriate fermion bilinears and determine the supergravity solution as in [5]; the supergravity solution is exactly as given in [5]. Note that the functions (G, H) are such that

$$e^H = y; \qquad z = \frac{1}{2}\tanh G, \tag{D.22}$$

where z is the defining function of the supergravity solution.

There is however a further issue in constructing the actual Killing spinors: the spinors by construction satisfy (D.4) and (D.5) since it is these equations which were processed. One still needs to check explicitly that all components of (D.6) are satisfied. Now the spinors as given in [5] do not depend at all on the time coordinate t. This is however inconsistent with the t component of (D.6); one can show that

$$\nabla_t \epsilon_a + M\gamma_t \epsilon_a \neq 0 \tag{D.23}$$

for constant (e_a^+, e_a^-). Another way to see that the Killing spinor solution is not quite correct is by considering the limiting case of $AdS_5 \times S^5$. The known explicit expressions for the Killing spinors of $AdS_5 \times S^5$ do depend explicitly on the time coordinate; this remains true for the specific combinations of spinors which form the set of sixteen discussed above.

The resolution of this issue is straightforward: (e_a^+, e_a^-) are not constant, but must contain suitable t (and indeed also ϕ) dependent phase factors so that (D.6) is satisfied. These phase factors drop out of (D.19) and all other fermion bilinears used to construct the supergravity solution.

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